Natural measures of diffeomorphisms with arbitrary Liouvillean rotation number

Philipp Kunde

Abstract

We construct smooth diffeomorphisms on the disc $\mathbb{D}^2$ and the annulus $\mathbb{S}^1 \times [0,1]$ with exactly three ergodic invariant measures and prescribed rotation number on the boundary. Moreover, these diffeomorphisms admit an invariant measurable Riemannian metric and are weak mixing with respect to the Lebesgue measure on the manifold.

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Introduction

By the well-known Brouwer fixed-point theorem every continuous function on the disc $\mathbb{D}^2$ has a fixed point. Indeed Bourgin proved with the aid of the Brouwer translation theorem that for every area-preserving orientation-preserving homeomorphism of the disc there is a fixed point inside the disc ([Bo68]). Hence any area- and orientation-preserving diffeomorphism of the disc has at least three ergodic invariant measures: The Dirac-measure δ at a fixed point in the interior of the disc, a measure supported at the boundary and any ergodic component of the area. In [FK04], §3, Fayad and Katok constructed diffeomorphisms with this minimal number of ergodic invariant measures. In fact they proved that the set of such diffeomorphisms is a residual subset in the closure $\mathcal{A}'\left(\mathbb{D}^2\right)$ in the C^∞-topology of the conjugates of rotations with conjugacies fixing every point of the boundary and the fixed points of the action by rotations (the boundary points and the fixed points of the action are called *singularities*).
As noted in [FK04] the pictures of rotations and conjugacies are essentially identical on the disc $\mathbb{D}^2$ and the annulus $\mathbb{S}^1 \times [0,1]$: We have polar coordinates (θ, r) and the rotations of the standard circle action $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ are given by $R_t(\theta, r) = (\theta + t, r)$. In this connection the origin of the disc, which is a fixed point of the circle action, corresponds to the boundary $\mathbb{S}^1 \times \{0\}$ in the case of the annulus (so considering the ergodic invariant measures the δ-measure at the fixed point of the circle action in the disc-case corresponds to the Lebesgue measure on the boundary component $\mathbb{S}^1 \times \{0\}$). Since all the conjugation maps of our constructions will coincide with the identity near $r = 0$ and $r = 1$ the differences between the disc and the annulus are insignificant. For the sake of convenience we will present our constructions in case of the annulus $\mathbb{S}^1 \times [0,1]$.
In both cases the Lebesgue measure μ on the manifold, the δ-measures at the fixed points of the rotations and the Lebesgue measures on the boundary components are called the *natural measures.*
We will extend the result of [FK04] by constructing diffeomorphisms with the minimal number of ergodic invariant measures in the restricted space

$$\mathcal{A}'_\alpha(M) := \overline{\left\{H \circ R_\alpha \circ H^{-1} \ : \ H \in \mathrm{Diff}^\infty(M,\mu),\ H = id \text{ on the singularities}\right\}}^{C^\infty}$$

for every Liouvillean number $\alpha \in \mathbb{S}^1$. In addition our constructed diffeomorphisms are weak mixing with respect to the area and preserve a measurable Riemannian metric. So this result is in line with [Kun13a], [Kun13b] and [Kun13c], where in extension of [GK00] constructions of diffeomorphisms with ergodic properties that preserve a measurable Riemannian measure are exhibited. At this juncture in [Kun13b] and [Kun13c] the number of ergodic invariant measures for diffeomorphisms on the torus $\mathbb{T}^m$ of dimension $m \geq 2$ is examined. By [Kun13b], Theorem 1, the set of weak mixing and strictly ergodic diffeomorphisms is a dense G_δ-set in $\mathcal{A}_\alpha(\mathbb{T}^m) = \overline{\{h \circ R_\alpha \circ h^{-1} : h \in Diff^\infty(\mathbb{T}^m,\mu)\}}^{C^\infty}$ for every Liouvillean number α. However, other numbers of ergodic invariant measures are possible as well: According to [Kun13c], Theorem 1, for any $d \in \mathbb{N}$ the set of minimal diffeomorphisms preserving exactly d ergodic measures and a measurable Riemannian metric is dense in $\mathcal{A}_\alpha(\mathbb{T}^m)$. The second result is connected to [Win01], where for any $d \in \mathbb{N}$ A. Windsor constructed minimal diffeomorphisms with d ergodic invariant measures in $\mathcal{A}(M) := \overline{\{h \circ S_t \circ h^{-1} \ : \ h \in \mathrm{Diff}^\infty(M,\nu),\, t \in \mathbb{S}^1\}}^{C^\infty}$ on any compact and connected smooth boundaryless manifold of dimension $m \geq 2$ admitting a free C^∞-action $\mathcal{S} = \{S_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume ν.
In this paper we consider the manifolds $\mathbb{D}^2$ and $\mathbb{S}^1 \times [0,1]$ with boundary. Indeed we will prove:

Theorem 1. *Let M be the disc $\mathbb{D}^2$ or the annulus $\mathbb{S}^1 \times [0,1]$ and $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ be the respective standard action by rotations. Then there exists a smooth diffeomorphism $f \in \mathcal{A}'_\alpha(M)$ that has* *exactly three ergodic invariant measures, namely the natural measures on M, is weak mixing with respect to the Lebesgue measure on M and preserves a measurable Riemannian metric.*

In section 1.2 we will conclude

Corollary 1. *Let M be the disc $\mathbb{D}^2$ or the annulus $\mathbb{S}^1 \times [0,1]$ and $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ be the respective standard action by rotations. Then the set of smooth diffeomorphisms $f \in \mathcal{A}'_\alpha(M)$ that have exactly three ergodic invariant measures, namely the natural measures on M, are weak mixing with respect to the Lebesgue measure on M and preserve a measurable Riemannian metric is a dense subset of $\mathcal{A}'_\alpha(M)$ in the C^∞-topology.*

as well as

Corollary 2. *The set of smooth diffeomorphisms $f \in \mathcal{A}'_\alpha(M)$ that have exactly three ergodic invariant measures, namely the natural measures on M, and are weak mixing with respect to the Lebesgue measure on M is a residual set (i.e. it contains a dense G_δ-set) in the C^∞-topology in $\mathcal{A}'_\alpha(M)$.*

1 Preliminaries

1.1 Definitions and notations

In addition to the definitions presented in [Kun13a], chapter 1.1., we introduce the subsequent notations:

Definition 1.1. 1. For a continuous function $F : [0,1] \times [-1,2] \to \mathbb{R}$

$$\|F\|_{0,\text{ext}} := \max_{z\in[0,1]\times[-1,2]} |F(z)|$$

2. Let $f \in \text{Diff}^k\left(\mathbb{S}^1 \times [-1,2]\right)$ with coordinate functions f_i be given. Then we consider f_i as a function $[0,1] \times [-1,2] \to \mathbb{R}$ and define

$$\|Df\|_{0,\text{ext}} := \max_{i,j\in\{1,2\}} \|D_j f_i\|_{0,\text{ext}}$$

and

$$|||f|||_{k,\text{ext}} := \max\left\{\|D_{\vec{a}} f_i\|_{0,\text{ext}}, \left\|D_{\vec{a}}\left(f_i^{-1}\right)\right\|_{0,\text{ext}} \;:\; i = 1,2,\ \vec{a} \text{ with } 0 \leq |\vec{a}| \leq k\right\}$$

1.2 Proof of the Corollaries

The main Theorem follows from the subsequent Proposition:

Proposition 1.2. *For every Liouvillean number α there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of rational numbers $\alpha_n = \frac{p_n}{q_n}$ satisfying $\lim_{n\to\infty} |\alpha - \alpha_n| = 0$ monotonically and a sequence $(h_n)_{n\in\mathbb{N}}$ of measure-preserving diffeomorphisms satisfying $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$ as well as $h_n = id$ in a neighbourhood of the boundary, such that the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ with $H_n = h_1 \circ h_2 \circ \ldots \circ h_n$ converge in the Diff$^\infty(M)$-topology and the diffeomorphism $f = \lim_{n\to\infty} f_n$ has exactly three ergodic invariant measures (namely the Lebesgue measure μ on $M = \mathbb{S}^1 \times [0,1]$, the Lebesgue measures δ^0 and δ^1 on the boundary components $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$ respectively), is weak mixing with respect to μ, admits an invariant measurable Riemannian metric and satisfies $f \in \mathcal{A}'_\alpha(M)$. Furthermore for every $\varepsilon > 0$ the parameters in the construction can be chosen in such a way that $d_\infty(f, R_\alpha) < \varepsilon$.*

By this Proposition weak mixing diffeomorphisms preserving exactly three ergodic measures as well as a measurable Riemannian metric are dense in $\mathcal{A}'_\alpha(M)$:
Because of $\mathcal{A}'_\alpha(M) = \overline{\{h \circ R_\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M,\mu),\ h = id \text{ on the boundary}\}}^{C^\infty}$ it is enough to show that for every diffeomorphism $h \in \text{Diff}^\infty(M,\mu)$, $h = id$ on the boundary, and every $\epsilon > 0$ there is a weak mixing diffeomorphism $\tilde{f}$ preserving a measurable Riemannian metric such that $d_\infty\left(\tilde{f}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. For this purpose let $h \in \text{Diff}^\infty(M,\mu)$ with $h = id$ on the boundary and $\epsilon > 0$ be arbitrary. By [Om74], p. 3, resp. [KM97], Theorem 43.1., $\text{Diff}^\infty(M)$ is a Lie group. In particular the conjugating map $g \mapsto h \circ g \circ h^{-1}$ is continuous with respect to the metric d_∞. Continuity in the point R_α yields the existence of $\delta > 0$, such that $d_\infty(g, R_\alpha) < \delta$ implies $d_\infty\left(h \circ g \circ h^{-1}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. By Proposition 1.2 we can find a weak mixing diffeomorphism f with exactly three ergodic invariant measures, f-invariant measurable Riemannian metric ω and $d_\infty(f, R_\alpha) < \delta$. Hence $\tilde{f} := h \circ f \circ h^{-1}$ satisfies $d_\infty\left(\tilde{f}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. Note that $\tilde{f}$ is weak mixing, has exactly three ergodic measures and preserves the measurable Riemannian metric $\tilde{\omega} := \left(h^{-1}\right)^* \omega$.
Hence Corollary 1 is deduced from Proposition 1.2.

Moreover, we can show that the set of weak mixing diffeomorphisms is generic in $\mathcal{A}'_\alpha(M)$ (i.e. it is a dense G_δ-set) using Proposition 1.2 and the same technique as in [Ha56], section *Category*, as well as [Kun13a], Remark 1.9..
Next let Ξ be a countable dense subset of $C(M,\mathbb{R})$. For $\rho \in \Xi$ and $\varepsilon > 0$ we consider the set

$$S(\rho,\varepsilon) :=$$
$$\left\{ f \in \mathcal{A}'_\alpha(M) \ : \ \exists N \in \mathbb{N} : \inf_{\xi \in \Theta} \left| \frac{1}{m} \sum_{i=0}^{m-1} \rho\left(f^i(x)\right) - \int_M \rho \, d\xi \right| < \varepsilon \text{ for every } m \geq N \text{ and } x \in M \right\}$$

at which Θ is the simplex generated by the measures μ, δ^0 and δ^1. Obviously such a set $S(\rho,\varepsilon)$ is open. It is also a dense subset of $\mathcal{A}'_\alpha(M)$ because every constructed diffeomorphism $f \in \mathcal{A}'_\alpha(M)$ is an element of $S(\rho,\varepsilon)$ due to Lemma 4.3 and the set of constructed diffeomorphisms is dense as seen above. By the same reasoning as at the end of section 4.1

$$\bigcap_{i \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} S\left(\rho_i, \frac{1}{k}\right),$$

which as a countable intersection of open and dense sets is a dense G_δ-set, is contained in the set of diffeomorphisms $f \in \mathcal{A}'_\alpha(M)$ with the natural measures as the only ergodic invariant measures. Since the intersection of dense G_δ-sets is a dense G_δ-set Corollary 2 is proven.

1.3 Sketch of the proof

The constructions are based on the "approximation by conjugation"-method developed by D.V. Anosov and A. Katok in [AK70]. Here one constructs successively a sequence of measure-preserving diffeomorphisms $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where the conjugation maps $H_n = h_1 \circ ... \circ h_n$ and the rational numbers $\alpha_n = \frac{p_n}{q_n}$ are chosen in such a way that the functions f_n converge to a diffeomorphism f with the aimed properties. Indeed we have to prove convergence of $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}'_\alpha\left(\mathbb{S}^1 \times [0,1]\right)$ for a prescribed Liouville number α. For it we need careful estimates on the norms of our explicitly defined conjugation maps in section 3.1.
In our setting the conjugation map h_n is made up of three maps introduced in section 2: $h_n = g_n \circ D^{-1}_{\psi_n,\gamma_n} \circ \phi_n \circ D_{\psi_n,\gamma_n}$, which coincides with the identity in a neighbourhood of the

boundary.
At this juncture the trapping map D_{ψ_n,γ_n} is used to gain control of almost everything of every orbit $\left\{H_n \circ R^k_{\alpha_{n+1}}(x)\right\}_{k=0,\dots,q_{n+1}-1}$ with the aid of the trapping regions. This allows us to prove a convergence result on Birkhoff sums (see Lemma 4.3), which in turn enables us to exclude the existence of further ergodic invariant measures besides the natural measures.
The conjugation map ϕ_n is used to map the trapping regions (which have nearly full length in the r-coordinate) on sets of small diameter and contrariwise to map elements of a partial partition η_n on stripes with r-length almost 1. The second property is used in the proof of the weak mixing property which is based on the notion of a (γ, ε)-distribution. In this proof we also need a map introducing shear in the θ-coordinate. The map g_n has to play this role. Since the conjugation maps have to act as an isometry on large parts of the manifold in order to construct a f-invariant measurable Riemannian metric a careful design of each conjugation map is required. The application of $D^{-1}_{\psi_n,\gamma_n}$ is necessary to make h_n to a diffeomorphism onto $\mathbb{S}^1 \times [0,1]$ (see Remark 2.12). Then we will construct the f-invariant measurable Riemannian metric by the same approach as in [Kun13a]: The conjugation maps are constructed in such a way that they act as isometries on elements of a partial partition ζ_n with respect to the standard metric ω_0. Since these partial partitions converge to the decomposition into points we can prove the convergence of the Riemannian metrics $\omega_n := \left(H_n^{-1}\right)^* \omega_0$ to a f-invariant measurable Riemannian metric.

2 Explicit constructions

Let $r(n) := r(n) = 8 \cdot n \cdot (n+5)$ and we put $\varepsilon_n := \frac{1}{4 \cdot n^{11} \cdot q_{n-1}^{5 \cdot r(n-1)+1}}$. In Remark 4.1 we will explain this choice of ε_n. Moreover, $\sigma_n \in (0,1)$ is a parameter that will be determined in Remark 3.12. Furthermore, we fix an arbitrary countable set $\Xi = \{\rho_1, \rho_2, ...\}$ of Lipschitz continuous functions $\rho_i : \mathbb{S}^1 \times [0,1] \to \mathbb{R}$ that is dense in $C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$. Since $C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$ is separable and Lipschitz continuous functions are dense in $C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$ this is possible. This set Ξ will be used in section 4.1 to prove that the *natural measures* are the only ergodic invariant ones.

2.1 The trapping map

To exclude the existence of further ergodic measures we have to gain control over a large proportion of the orbit $\left\{H_n \circ R^i_{\alpha_{n+1}}(x)\right\}_{i=0,1,\dots,q_{n+1}-1}$ for every $x \in \mathbb{S}^1 \times [0,1]$. For this purpose we use for every $n \in \mathbb{N}$ a smooth map $\psi_n : [0,1] \to \mathbb{R}$ satisfying

- ψ_n is non-decreasing on $\left[0, \frac{1}{2}\right]$ and non-increasing on $\left[\frac{1}{2}, 1\right]$.
- ψ_n is equal to $k \cdot 4\varepsilon_n$ on $\left[\frac{k}{n^2} + \frac{1}{n^4}, \frac{k+1}{n^2} - \frac{1}{n^4}\right]$ for $0 \le k \le \left\lfloor \frac{n^2}{2} \right\rfloor - 1$ and ψ_n is equal to $k \cdot 4\varepsilon_n$ on $\left[\frac{n^2-k-1}{n^2} + \frac{1}{n^4}, \frac{n^2-k}{n^2} - \frac{1}{n^4}\right]$ for $0 \le k \le \left\lfloor \frac{n^2}{2} \right\rfloor - 1$. On $\left[\frac{\left\lfloor \frac{n^2}{2} \right\rfloor - 1}{n^2}, \frac{n^2 - \left\lfloor \frac{n^2}{2} \right\rfloor}{n^2}\right]$ it is put to $\left(\left\lfloor \frac{n^2}{2} \right\rfloor - 1\right) \cdot 4\varepsilon_n$.

With it we define the map $\overline{D}_{\psi_n} : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ by:

$$(\theta, r) \mapsto \left(\theta, r + \left(1 + \frac{1}{q_n^3} + \frac{1}{q_n^4} + ... + \frac{1}{q_n^{3+n-1}}\right) \cdot \psi_n(\theta)\right).$$

Using the maps $C_{\gamma_n}(\theta, r) = (\gamma_n \cdot \theta, r)$ we construct the map

$$D_{\psi_n,\gamma_n} \coloneqq C_{\gamma_n}^{-1} \circ \overline{D}_{\psi_n} \circ C_{\gamma_n} : \quad \left[0, \frac{1}{\gamma_n}\right] \times \mathbb{R} \to \left[0, \frac{1}{\gamma_n}\right] \times \mathbb{R}.$$

Since this map coincides with the identity in a neighbourhood of the boundary of the sector on the θ-axis we can extend it to a smooth map $D_{\psi_n,\gamma_n} : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ using the description $D_{\psi_n,\gamma_n} \circ R_{\frac{l}{\gamma_n}} = R_{\frac{l}{\gamma_n}} \circ D_{\psi_n,\gamma_n}$ for any $l \in \mathbb{Z}$. In our construction we use

$$\gamma_n = n \cdot q_n^{2+3+4+...+(3+n-1)} = n \cdot q_n^{2+3\cdot n + \frac{n\cdot(n-1)}{2}}$$

Remark 2.1. The trapping map D_{ψ_n,γ_n} causes a r-translation by at most $2 \cdot \left(\left\lfloor \frac{n^2}{2} \right\rfloor - 1 \right) \cdot 4\varepsilon_n \leq 4n^2 \cdot \varepsilon_n$.

Remark 2.2. We have $D_{\psi_n,\gamma_n}\left(\mathbb{S}^1 \times [0,1]\right) \subset \mathbb{S}^1 \times [-1,2]$. This motivates our definition of $\|\cdot\|_{0,\text{ext}}$ and is used in the norm estimates in section 3.1 implicitly.

2.2 Trapping regions

We introduce three kind of trapping regions:
In the interior of $\mathbb{S}^1 \times [0,1]$ and for $l \in \mathbb{Z}$ as well as $k = 0, ..., n-1$ we consider the sets

$$\begin{aligned}
&S^{\text{int}}_{l,k,j_1^{(1)},\vec{j_2}} = \\
&\bigcup \Bigg[\frac{l}{q_n} + \frac{k}{n\cdot q_n} + \frac{j_1^{(1)}}{n\cdot q_n^2} + \frac{t_1^{(1)}}{n\cdot q_n^3} + ... + \frac{t_1^{\left(3\cdot k + \frac{k\cdot(k-1)}{2}\right)}}{n\cdot q_n^{2+3+4+...+(3+k-1)}} + \frac{j_2^{(1)}}{n\cdot q_n^{2+3+4+...+(3+k-1)+1}} + ... \\
&\quad + \frac{j_2^{(3+k)}}{n\cdot q_n^{2+3+4+...+(3+k-1)+3+k}} + \frac{t_1^{\left(3\cdot k + \frac{k\cdot(k-1)}{2}+1\right)}}{n\cdot q_n^{2+3+...+(3+k-1)+(3+k)+1}} + ... + \frac{t_1^{\left(3\cdot(n-1) + \frac{n\cdot(n-1)}{2} - k\right)}}{\gamma_n} + \frac{1}{n^4\cdot\gamma_n}, \\
&\quad \frac{l}{q_n} + \frac{k}{n\cdot q_n} + ... + \frac{t_1^{\left(3\cdot(n-1) + \frac{n\cdot(n-1)}{2} - k\right)} + 1}{\gamma_n} - \frac{1}{n^4\cdot\gamma_n}\Bigg] \\
&\quad \times \left[\frac{t_2^{(1)}}{q_n} + ... + \frac{t_2^{(3+k)}}{q_n^{3+k}} + \frac{\varepsilon_n}{q_n^{3+k}}, \frac{t_2^{(1)}}{q_n} + ... + \frac{t_2^{(3+k)}+1}{q_n^{3+k}} - \frac{\varepsilon_n}{q_n^{3+k}} \right]
\end{aligned}$$

where the union is taken over $t_1^{(j)} \in \mathbb{Z}$, $0 \leq t_1^{(j)} \leq q_n - 1$, for $j = 1, ..., 3\cdot(n-1) + \frac{n\cdot(n-1)}{2} - k$ apart from $t_1^{\left(3\cdot k + \frac{k\cdot(k-1)}{2}+1\right)}$ satisfying $\lceil \varepsilon_n \cdot q_n \rceil \leq t_1^{\left(3\cdot k + \frac{k\cdot(k-1)}{2}+1\right)} \leq q_n - \lceil \varepsilon_n \cdot q_n \rceil - 1$ as well as $t_2^{(1)} \in \mathbb{Z}$, $\left\lceil \left(4n^2+1\right)\varepsilon_n \cdot q_n \right\rceil \leq t_2^{(1)} \leq q_n - \left\lceil \left(4n^2+1\right)\varepsilon_n \cdot q_n \right\rceil - 1$ as well as $t_2^{(l)} \in \mathbb{Z}$, $0 \leq t_2^{(l)} \leq q_n - 1$, for $l = 2, ..., 3+k$.
Then the set of trapping regions of the first kind consists of all sets $D^{-1}_{\psi_n,\gamma_n}\left(S^{\text{int}}_{l,k,j_1^{(1)},\vec{j_2}}\right)$, where all $j_i^{(1)} \in \mathbb{Z}$ satisfy $\lceil 18n^2\varepsilon_n \cdot q_n \rceil \leq j_i^{(1)} \leq q_n - \lceil 18n^2\varepsilon_n \cdot q_n \rceil - 1$ for $i = 1, 2$ and $j_2^{(s)} \in \mathbb{Z}$, $0 \leq j_2^{(s)} \leq q_n - 1$ for $s = 2, ..., 3+k$.

In the neighbourhood of the boundary $\mathbb{S}^1 \times \{0\}$ we introduce the trapping regions of the second kind $\tilde{S}^0_{l,k,j_1^{(1)},\vec{j_2}} \coloneqq D^{-1}_{\psi_n,\gamma_n}\left(S^0_{l,k,j_1^{(1)},\vec{j_2}}\right) \cap \left(\mathbb{S}^1 \times [0,1]\right)$, at which

$$S^0_{l,k,j_1^{(1)},\vec{j_2}} =$$

$$\bigcup \Bigg[\frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2} + \frac{t_1^{(1)}}{n \cdot q_n^3} + ... + \frac{t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2}\right)}}{n \cdot q_n^{2+3+4+...+(3+k-1)}} + \frac{j_2^{(1)}}{n \cdot q_n^{2+3+4+...+(3+k-1)+1}} + ...$$
$$+ \frac{j_2^{(3+k)}}{n \cdot q_n^{2+3+4+...+(3+k-1)+3+k}} + \frac{t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)}}{n \cdot q_n^{2+3+...+(3+k-1)+(3+k)+1}} + ... + \frac{t_1^{\left(3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k\right)}}{\gamma_n} + \frac{1}{n^4 \cdot \gamma_n},$$
$$\frac{l}{q_n} + \frac{k}{n \cdot q_n} + ... + \frac{t_1^{\left(3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k\right)} + 1}{\gamma_n} - \frac{1}{n^4 \cdot \gamma_n} \Bigg]$$
$$\times \left[0, 4 \cdot n^2 \cdot \varepsilon_n\right]$$

where the union is taken over all $t_1^{(j)} \in \mathbb{Z}$, $0 \leq t_1^{(j)} \leq q_n - 1$, for $j = 1, ..., 3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k$ apart from $t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)}$ satisfying $\lceil \varepsilon_n \cdot q_n \rceil \leq t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)} \leq q_n - \lceil \varepsilon_n \cdot q_n \rceil - 1$.
Then the set of trapping regions of the second kind consists of all sets $\tilde{S}^0_{l,k,j_1^{(1)},\vec{j_2}}$, where all $j_i^{(1)} \in \mathbb{Z}$ satisfy $\lceil 18n^2 \varepsilon_n \cdot q_n \rceil \leq j_i^{(1)} \leq q_n - \lceil 18n^2 \varepsilon_n \cdot q_n \rceil - 1$ for $i = 1, 2$ and $j_2^{(s)} \in \mathbb{Z}$, $0 \leq j_2^{(s)} \leq q_n - 1$ for $s = 2, ..., 3 + k$.

In the neighbourhood of the boundary $\mathbb{S}^1 \times \{1\}$ we introduce the trapping regions of the third kind $\tilde{S}^1_{l,k,j_1^{(1)},\vec{j_2}} := D^{-1}_{\psi_n,\gamma_n}\left(S^1_{l,k,j_1^{(1)},\vec{j_2}}\right) \cap \left(\mathbb{S}^1 \times [0,1]\right)$, at which

$$S^1_{l,k,j_1^{(1)},\vec{j_2}} =$$

$$\bigcup \Bigg[\frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2} + \frac{t_1^{(1)}}{n \cdot q_n^3} + ... + \frac{t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2}\right)}}{n \cdot q_n^{2+3+4+...+(3+k-1)}} + \frac{j_2^{(1)}}{n \cdot q_n^{2+3+4+...+(3+k-1)+1}} + ...$$
$$+ \frac{j_2^{(3+k)}}{n \cdot q_n^{2+3+4+...+(3+k-1)+3+k}} + \frac{t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)}}{n \cdot q_n^{2+3+...+(3+k-1)+(3+k)+1}} + ... + \frac{t_1^{\left(3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k\right)}}{\gamma_n} + \frac{1}{n^4 \cdot \gamma_n},$$
$$\frac{l}{q_n} + \frac{k}{n \cdot q_n} + ... + \frac{t_1^{\left(3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k\right)} + 1}{\gamma_n} - \frac{1}{n^4 \cdot \gamma_n} \Bigg]$$
$$\times \left[1 - 4 \cdot n^2 \cdot \varepsilon_n, 1\right]$$

where the union is taken over all $t_1^{(j)} \in \mathbb{Z}$, $0 \leq t_1^{(j)} \leq q_n - 1$, for $j = 1, ..., 3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k$ apart from $t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)}$ satisfying $\lceil \varepsilon_n \cdot q_n \rceil \leq t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)} \leq q_n - \lceil \varepsilon_n \cdot q_n \rceil - 1$.
Then the set of trapping regions of the third kind consists of all sets $\tilde{S}^1_{l,k,j_1^{(1)},\vec{j_2}}$, where all $j_i^{(1)} \in \mathbb{Z}$ satisfy $\lceil 18n^2 \varepsilon_n \cdot q_n \rceil \leq j_i^{(1)} \leq q_n - \lceil 18n^2 \varepsilon_n \cdot q_n \rceil - 1$ for $i = 1, 2$ and $j_2^{(s)} \in \mathbb{Z}$, $0 \leq j_2^{(s)} \leq q_n - 1$ for $s = 2, ..., 3 + k$.

Remark 2.3. By the requirements on the numbers $t_1^{(u)}$ and $j_i^{(u)}$ all blocks overlying $\frac{1}{\gamma_n}$-sections on the θ-axis, that are part of trapping regions belonging to one kind are also part of trapping regions belonging to the other kinds.
Let $x = (\theta, r) \in \mathbb{S}^1 \times [0,1]$ be arbitrary. By the construction of the map $\bar{D}_{\psi_n}$ there are at

most four sections $\left[\frac{k}{n^2}+\frac{1}{n^4},\frac{k+1}{n^2}-\frac{1}{n^4}\right]$ on the domain $[0,1]$ such that r does not belong to either $\psi_n^{-1}\left(\left[\frac{k}{n^2}+\frac{1}{n^4},\frac{k+1}{n^2}-\frac{1}{n^4}\right]\times\left[0,4n^2\cdot\varepsilon_n\right]\right)$, $\psi_n^{-1}\left(\left[\frac{k}{n^2}+\frac{1}{n^4},\frac{k+1}{n^2}-\frac{1}{n^4}\right]\times\left[1-4n^2\cdot\varepsilon_n,1\right]\right)$ or $\psi_n^{-1}\left(\left[\frac{k}{n^2}+\frac{1}{n^4},\frac{k+1}{n^2}-\frac{1}{n^4}\right]\times\left[\left(4n^2+1\right)\cdot\varepsilon_n,1-\left(4n^2+1\right)\cdot\varepsilon_n\right]\right)$.
We have to bear the gaps of our trapping region in the r-coordinate in mind. Therefore we note that $\left(1+\frac{1}{q_n^3}+...+\frac{1}{q_n^{3+k-1}}\right)\cdot 4\varepsilon_n$ is a multiple of $\frac{1}{q_n^{3+k}}$ and this translates by full $\frac{1}{q_n^{3+k}}$-blocks in the r-coordinate. Hence there are at most four further sections $\left[\frac{k}{n^2\gamma_n}+\frac{1}{n^4\gamma_n},\frac{k+1}{n^2\gamma_n}-\frac{1}{n^4\gamma_n}\right]$ on $\left[0,\frac{1}{\gamma_n}\right]$ such that r does not belong to either $D^{-1}_{\psi_n,\gamma_n}\left(\left[\frac{k}{n^2\gamma_n}+\frac{1}{n^4\gamma_n},\frac{k+1}{n^2\gamma_n}-\frac{1}{n^4\gamma_n}\right]\times\left[0,4n^2\cdot\varepsilon_n\right]\right)$ or $D^{-1}_{\psi_n,\gamma_n}\left(\left[\frac{k}{n^2\gamma_n}+\frac{1}{n^4\gamma_n},\frac{k+1}{n^2\gamma_n}-\frac{1}{n^4\gamma_n}\right]\times\left[\frac{t_2^{(1)}}{q_n}+...+\frac{t_2^{(3+k)}}{q_n^{3+k}}+\frac{\varepsilon_n}{q_n^{3+k}},\frac{t_2^{(1)}}{q_n}+...+\frac{t_2^{(3+k)}+1}{q_n^{3+k}}-\frac{\varepsilon_n}{q_n^{3+k}}\right]\right)$ or $D^{-1}_{\psi_n,\gamma_n}\left(\left[\frac{k}{n^2\gamma_n}+\frac{1}{n^4\gamma_n},\frac{k+1}{n^2\gamma_n}-\frac{1}{n^4\gamma_n}\right]\times\left[1-4n^2\cdot\varepsilon_n,1\right]\right)$.
For $l=0,...,q_n-1$, $k=0,1,...,n-1$ a trapping region on $\left[\frac{l}{q_n}+\frac{k}{n\cdot q_n},\frac{l}{q_n}+\frac{k+1}{n\cdot q_n}\right]\times[0,1]$ consists of at least $\left(1-3\cdot\varepsilon_n\right)\cdot q_n^{3n+\frac{n\cdot(n-1)}{2}-(3+k)}$ many $\frac{1}{\gamma_n}$-sections. We fix $l,k,j_1^{(1)},\vec{j_2}$. Since $\{i\cdot\alpha_{n+1}\}_{i=0,...,q_{n+1}-1}$ is equidistributed on $\mathbb{S}^1$ the number of iterates i, such that the orbit $\left\{R^i_{\alpha_{n+1}}(x)\right\}_{i=0,...,q_{n+1}-1}$ is captured by one of the 3 trapping regions $D^{-1}_{\psi_n,\gamma_n}\left(S^t_{l,k,j_1^{(1)},\vec{j_2}}\right)\cap\left(\mathbb{S}^1\times[0,1]\right)$, $t\in\{\text{int},0,1\}$, is at least

$$\left(1-3\cdot\varepsilon_n\right)\cdot q_n^{3n+\frac{n\cdot(n-1)}{2}-(3+k)}\cdot\left(n^2-8\right)\cdot\left\lfloor q_{n+1}\cdot\frac{1-\frac{2}{n^2}}{n^2\cdot\gamma_n}\right\rfloor.$$

Depending on the point $x\in\mathbb{S}^1\times[0,1]$ there is a portion $\varpi^n_t(x)$ of these iterates spent in trapping regions of the specific kind, $t\in\{\text{int},0,1\}$. This portion does not depend on the indices $l,k,j_1^{(1)},\vec{j_2}$. Then the number of iterates i, such that the orbit $\left\{R^i_{\alpha_{n+1}}(x)\right\}_{i=0,...,q_{n+1}-1}$ meets an arbitrary trapping region $D^{-1}_{\psi_n,\gamma_n}\left(S^t_{l,k,j_1^{(1)},\vec{j_2}}\right)\cap\left(\mathbb{S}^1\times[0,1]\right)$, is not less than

$$\begin{aligned}&q_n^{3n+\frac{n\cdot(n-1)}{2}-(3+k)}\cdot\varpi^n_t(x)\cdot\left(n^2-8\right)\cdot q_{n+1}\cdot\frac{1-\frac{4}{n^2}}{n^2\cdot\gamma_n}\\ \geq&\varpi^n_t(x)\cdot q_{n+1}\cdot\left(n^2-8\right)\cdot\frac{1-\frac{4}{n^2}}{n^3\cdot q_n^{2+3+k}}\\ \geq&\varpi^n_t(x)\cdot q_{n+1}\cdot\left(1-\frac{12}{n^2}\right)\cdot\frac{1}{n\cdot q_n^{5+k}}\end{aligned}$$

iterates. Moreover, for every $t\in\{\text{int},0,1\}$ there are $\left(q_n-2\cdot\lceil 18n^2\varepsilon_n\cdot q_n\rceil\right)^2\cdot q_n^{2+k}$ trapping regions of the specific kind on $\left[\frac{l}{q_n}+\frac{k}{n^2\cdot q_n},\frac{l}{q_n}+\frac{k+1}{n^2\cdot q_n}\right]\times N_t\times\mathbb{T}^{m-2}$ for $l=0,...,q_n-1$ as well as $k=0,...,n-1$ and so not less than

$$\begin{aligned}&\left(q_n-2\cdot\lceil 18n^2\varepsilon_n\cdot q_n\rceil\right)^2\cdot q_n^{2+k}\cdot q_{n+1}\cdot\left(1-\frac{12}{n^2}\right)\cdot\frac{1}{n\cdot q_n^{2+(3+k)}}\\ \geq& q_{n+1}\cdot\left(1-\frac{1}{n^6}\right)^2\cdot\left(1-\frac{12}{n^2}\right)\cdot\frac{1}{n\cdot q_n}\geq q_{n+1}\cdot\left(1-\frac{14}{n^2}\right)\cdot\frac{1}{n\cdot q_n}\end{aligned}$$

iterates are trapped here. Altogether at least $q_{n+1}\cdot\left(1-\frac{14}{n^2}\right)$ iterates are captured.

Remark 2.4. On the contrary at most $\frac{14}{n^2} \cdot q_{n+1}$ iterates are not captured by the trapping regions.

2.3 Sequences of partial partitions

In this subsection we define the two announced sequences of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ of $M = \mathbb{S}^1 \times [0,1]$.

2.3.1 Partial partition η_n

Initially η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]$. For this purpose we divide the fundamental sector in n sections:

- On $\left[\frac{k}{n\cdot q_n}, \frac{k+1}{n\cdot q_n}\right] \times [0,1]$ in case of $k \in \mathbb{N}$ and $0 \leq k \leq n-2$ the partial partition η_n consists of all multidimensional intervals of the following form:

$$\left[\frac{k}{n\cdot q_n} + \frac{j_1^{(1)}}{n\cdot q_n^2} + ... + \frac{j_1^{\left(1+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}\right)}}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}} + \frac{18n^2\cdot\varepsilon_n}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}},\right.$$
$$\left.\frac{k}{n\cdot q_n} + \frac{j_1^{(1)}}{n\cdot q_n^2} + ... + \frac{j_1^{\left(1+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}\right)}+1}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}} - \frac{18n^2\cdot\varepsilon_n}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}}\right]$$
$$\times\left[\frac{j_2^{(1)}}{q_n} + ... + \frac{j_2^{(3+k+1)}}{q_n^{3+k+1}} + \frac{\varepsilon_n}{4\cdot q_n^{3+k+1}}, \frac{j_2^{(1)}}{q_n} + ... + \frac{j_2^{(3+k+1)}+1}{q_n^{3+k+1}} - \frac{\varepsilon_n}{4\cdot q_n^{3+k+1}}\right]$$

  where $j_2^{(l)} \in \mathbb{Z}$, $\lceil 18n^2\cdot\varepsilon_n\cdot q_n\rceil \leq j_2^{(l)} \leq q_n - \lceil 18n^2\cdot\varepsilon_n\cdot q_n\rceil - 1$ for $l = 1, ..., 3+k+1$ and $j_1^{(l)} \in \mathbb{Z}$, $\lceil 18n^2\cdot\varepsilon_n\cdot q_n\rceil \leq j_1^{(l)} \leq q_n - \lceil 18n^2\cdot\varepsilon_n\cdot q_n\rceil - 1$ for $l = 1, ..., 1+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}$.

- On $\left[\frac{n-1}{n\cdot q_n}, \frac{1}{q_n}\right] \times [0,1]$ there are no elements of the partial partition η_n.

As the image under R_{l/q_n} with $l \in \mathbb{Z}$ this partial partition of $\left[0, \frac{1}{q_n}\right] \times [0,1]$ is extended to a partial partition of $\mathbb{S}^1 \times [0,1]$.

Remark 2.5. By construction this sequence of partial partitions converges to the decomposition into points.

2.3.2 Partial partition ζ_n

As in the previous case we will construct the partial partition ζ_n on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]$ initially and therefore divide this sector into n sections:
On $\left[\frac{k}{n\cdot q_n}, \frac{k+1}{n\cdot q_n}\right] \times [0,1]$ in case of $k \in \mathbb{N}$ and $0 \leq k \leq n-1$ the partial partition ζ_n consists of all sets $\check{\Gamma}_n = D^{-1}_{\psi_n,\gamma_n}\left(\check{I}_n\right)$, where $\check{I}_n$ is a multidimensional interval of the following form:

$$\left[\frac{k}{n\cdot q_n}+\frac{j_1^{(1)}}{n\cdot q_n^2}+...+\frac{j_1^{\left(1+3\cdot n+\frac{n\cdot(n-1)}{2}\right)}}{\gamma_n}+\frac{s}{n^2\cdot\gamma_n}+\frac{1}{n^4\cdot\gamma_n},\right.$$
$$\left.\frac{k}{n\cdot q_n}+\frac{j_1^{(1)}}{n\cdot q_n^2}+...+\frac{j_1^{\left(1+3\cdot n+\frac{n\cdot(n-1)}{2}\right)}}{\gamma_n}+\frac{s+1}{n^2\cdot\gamma_n}-\frac{1}{n^4\cdot\gamma_n}\right]$$
$$\times\left[\frac{j_2^{(1)}}{q_n}+...+\frac{j_2^{(3+k+1)}}{q_n^{3+k+1}}+...+\frac{j_2^{\left(2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}\right)}}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}}+\right.$$
$$\frac{j_2^{\left(3+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}\right)}\cdot 16n^2\cdot\varepsilon_n}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}\cdot\lceil nq_n^{\sigma_n}\rceil}+\frac{1600n^4\cdot\varepsilon_n^2}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}\cdot\lceil nq_n^{\sigma_n}\rceil},$$
$$\left.\frac{j_2^{(1)}}{q_n}+...+\frac{\left(j_2^{\left(3+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}\right)}+1\right)\cdot 16n^2\cdot\varepsilon_n}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}\cdot\lceil nq_n^{\sigma_n}\rceil}-\frac{1600n^4\cdot\varepsilon_n^2}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}\cdot\lceil nq_n^{\sigma_n}\rceil}\right]$$

where $j_1^{(l)}\in\mathbb{Z}$ and $\lceil 100n^2\cdot\varepsilon_n\cdot q_n\rceil\le j_1^{(l)}\le q_n-\lceil 100n^2\cdot\varepsilon_n\cdot q_n\rceil-1$ for $l=1,...,1+3\cdot n+\frac{n\cdot(n-1)}{2}$, $j_2^{(l)}\in\mathbb{Z}$ and $\lceil 100n^2\cdot\varepsilon_n\cdot q_n\rceil\le j_2^{(l)}\le q_n-\lceil 100n^2\cdot\varepsilon_n\cdot q_n\rceil-1$ for $l=1,...,2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}$, $j_2^{\left(3+3(k+1)+\frac{k(k+1)}{2}\right)}\in\mathbb{Z}$, $\lceil nq_n^{\sigma_n}\rceil\cdot n\le j_2^{\left(3+3(k+1)+\frac{k(k+1)}{2}\right)}\le\frac{\lceil nq_n^{\sigma_n}\rceil}{16n\cdot\varepsilon_n}-\lceil nq_n^{\sigma_n}\rceil\cdot n-1$ as well as $s\in\mathbb{N}$ and $0\le s\le n^2-1$.

Remark 2.6. For every n the partial partition ζ_n consists of disjoint sets, covers a set of measure at least $1-\frac{3}{n^2}$ in case of $n\ge 3$ and the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points.

Remark 2.7. Note that D_{ψ_n,γ_n} acts as an isometry on all the partition elements $\check{\Gamma}_n\in\zeta_n$.

2.4 The conjugation map g_n

Let $a,b\in\mathbb{Z}$ and $\varepsilon\in\left(0,\frac{1}{16}\right]$ such that $\frac{1}{\varepsilon}\in\mathbb{Z}$. Moreover, we consider $\delta>0$, such that $\frac{1}{\delta}\in\mathbb{Z}$ and $\frac{a\cdot b\cdot\delta}{\varepsilon}\in\mathbb{Z}$. We denote $[0,1]^2$ by Δ and $[\varepsilon,1-\varepsilon]^2$ by $\Delta(\varepsilon)$. In this setting we recall [Kun13a], Lemma 2.4.:

Lemma 2.8. *For every $\varepsilon\in\left(0,\frac{1}{16}\right]$ there exists a smooth measure-preserving diffeomorphism $g_\varepsilon:[0,1]^2\to\{(x+\varepsilon\cdot y,y)\ :\ x,y\in[0,1]\}$, that is the identity on $\Delta(4\varepsilon)$ and coincides with the map $(x,y)\mapsto(x+\varepsilon\cdot y,y)$ on $\Delta\setminus\Delta(\varepsilon)$.*

Let $b\in\mathbb{Z}$, $\tilde{g}_b:\mathbb{S}^1\times[0,1]\to\mathbb{S}^1\times[0,1]$ be the smooth measure-preserving diffeomorphism given by $\tilde{g}_b(\theta,r)=(\theta+b\cdot r,r)$ and denote $\left[0,\frac{1}{a}\right]\times\left[0,\frac{\varepsilon}{b\cdot a}\right]$ by $\Delta_{a,b,\varepsilon}$. Using the map $D_{a,b,\varepsilon}:\mathbb{R}^2\to\mathbb{R}^2,(\theta,r)\mapsto\left(a\cdot\theta,\frac{b\cdot a}{\varepsilon}\cdot r\right)$ and g_ε from Lemma 2.8 we define the measure-preserving diffeomorphism $g_{a,b,\varepsilon}:\Delta_{a,b,\varepsilon}\to\tilde{g}_b(\Delta_{a,b,\varepsilon})$ by $g_{a,b,\varepsilon}=D^{-1}_{a,b,\varepsilon}\circ g_\varepsilon\circ D_{a,b,\varepsilon}$. Using the fact that $\frac{a\cdot b\cdot\delta}{\varepsilon}\in\mathbb{Z}$ we extend it to a smooth diffeomorphism $g_{a,b,\varepsilon,\delta}:\left[0,\frac{1}{a}\right]\times[\delta,1-\delta]\to\tilde{g}_b\left(\left[0,\frac{1}{a}\right]\times[\delta,1-\delta]\right)$ by the description:

$$g_{a,b,\varepsilon,\delta}\left(\theta,r+l\cdot\frac{\varepsilon}{b\cdot a}\right)=\left(l\cdot\frac{\varepsilon}{a},l\cdot\frac{\varepsilon}{b\cdot a}\right)+g_{a,b,\varepsilon}(\theta,r)$$

for $r\in\left[0,\frac{\varepsilon}{b\cdot a}\right]$ and $l\in\mathbb{Z}$ satisfying $\frac{\delta}{\varepsilon}\cdot b\cdot a\le l\le\frac{1-\delta}{\varepsilon}\cdot b\cdot a-1$.
With the choice $\delta=12n^2\cdot\varepsilon_n$ we construct the smooth measure-preserving diffeomorphism g_n

on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times \left[12n^2 \cdot \varepsilon_n, 1 - 12n^2 \cdot \varepsilon_n\right]$ initially and for this divide it into n sections:
On $\left[\frac{k}{n \cdot q_n}, \frac{k+1}{n \cdot q_n}\right] \times \left[12n^2 \cdot \varepsilon_n, 1 - 12n^2 \cdot \varepsilon_n\right]$ in case of $k \in \mathbb{Z}$ and $0 \leq k \leq n-1$:

$$g_n = g_{n \cdot q_n^{2+3\cdot(k+1)+\frac{(k+1)\cdot k}{2}}, [n \cdot q_n^{\sigma_n}], 16n^2 \cdot \varepsilon_n, 12n^2 \cdot \varepsilon_n}$$

Since g_n coincides with the map $\tilde{g}_{[n \cdot q_n^{\sigma_n}]}$ in a neighbourhood of the boundary of the different sections on the θ-axis this yields a smooth map and we can extend it to a smooth measure-preserving diffeomorphism on $\mathbb{S}^1 \times \left[12n^2 \cdot \varepsilon_n, 1 - 12n^2 \cdot \varepsilon_n\right]$ using the description $g_n \circ R_{\frac{l}{q_n}} = R_{\frac{l}{q_n}} \circ g_n$ for $l \in \mathbb{Z}$.
Moreover, let $\chi_n : [0,1] \to [0,1]$ be a smooth function satisfying the subsequent properties:

- χ_n is equal to 0 on $\left[0, 4n^2 \cdot \varepsilon_n\right]$ as well as on $\left[1 - 8n^2 \cdot \varepsilon_n, 1\right]$. On $\left[6n^2 \cdot \varepsilon_n, 1 - 10n^2 \cdot \varepsilon_n\right]$ χ_n takes the value 1.
- χ_n is non-decreasing on $\left[4n^2 \cdot \varepsilon_n, 6n^2 \cdot \varepsilon_n\right]$ and non-increasing on $\left[1 - 10n^2 \cdot \varepsilon_n, 1 - 8n^2 \cdot \varepsilon_n\right]$.

With it we define $g_n : \mathbb{S}^1 \times \left[0, 12n^2 \cdot \varepsilon_n\right] \to \mathbb{S}^1 \times \left[0, 12n^2 \cdot \varepsilon_n\right]$ and $g_n : \mathbb{S}^1 \times \left[1 - 12n^2 \cdot \varepsilon_n, 1\right] \to \mathbb{S}^1 \times \left[1 - 12n^2 \cdot \varepsilon_n, 1\right]$ by

$$g_n(\theta, r) = \left(\theta + \chi_n(r) \cdot [n \cdot q_n^{\sigma_n}] \cdot r, r\right)$$

Since all the constructed maps g_n coincide with $\tilde{g}_{[nq_n^{\sigma_n}]}$ in a neighbourhood of the boundary of the respective domain we can piece them together smoothly to a diffeomorphism $g_n : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$.
We note that the assumption $\frac{a \cdot b \cdot \delta}{\varepsilon} = \frac{a \cdot b \cdot 3}{4} \in \mathbb{Z}$ is satisfied, because $\frac{1}{\varepsilon_n} = 4 \cdot n^{11} \cdot q_{n-1}^{5 \cdot r(n-1)+1}$ divides q_n by our construction of the sequence $(\alpha_n)_{n \in \mathbb{N}}$ in Lemma 3.8. Moreover, $g_n = id$ in the neighbourhoods $\mathbb{S}^1 \times \left[0, 4n^2 \cdot \varepsilon_n\right]$ and $\mathbb{S}^1 \times \left[1 - 8n^2 \cdot \varepsilon_n, 1\right]$ of the boundary components.

Remark 2.9. We will call the parts of the domains $\Delta_{a,b,\varepsilon,\delta}$ corresponding to $\Delta(4\varepsilon)$ of g_ε the "good area" of g_n.

2.5 The conjugation map ϕ_n

We modify [Kun13a], Lemma 2.6.:

Lemma 2.10. *For every $j \in \mathbb{N}$ and $0 < \varepsilon < \frac{1}{4 \cdot j}$ there exists a smooth measure-preserving diffeomorphism $\varphi_{\varepsilon,j}$ on $\mathbb{R}^2$, which is the rotation in the plane by $\pi/2$ about the point $\left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$ on $[(j+1) \cdot \varepsilon, 1 - (j+1) \cdot \varepsilon]^2$ and coincides with the identity outside of $[j \cdot \varepsilon, 1 - j \cdot \varepsilon]^2$.*

Proof. First of all we introduce the notation $\Delta(\varepsilon) := [\varepsilon, 1 - \varepsilon]^2$. Let ψ_ε be a smooth diffeomorphism satisfying

$$\psi_\varepsilon(x, y) = \begin{cases} (x, y) & \textit{on } \mathbb{R}^2 \setminus \Delta(j \cdot \varepsilon) \\ \left(\frac{1}{2} + \frac{1}{5} \cdot \left(x - \frac{1}{2}\right), \frac{1}{2} + \frac{1}{5} \cdot \left(y - \frac{1}{2}\right)\right) & \textit{on } \Delta((j+1) \cdot \varepsilon) \end{cases}$$

Furthermore let τ_ε be a smooth diffeomorphism with the following properties

$$\tau_\varepsilon(x, y) = \begin{cases} (1 - y, x) & \textit{on } \left\{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \frac{1}{50}\right\} \\ (x, y) & \textit{on } \left\{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \geq \frac{1}{16}\right\} \end{cases}$$

We define $\bar{\varphi}_{\varepsilon} := \psi_{\varepsilon}^{-1} \circ \tau_{\varepsilon} \circ \psi_{\varepsilon}$. Then the diffeomorphism $\bar{\varphi}_{\varepsilon}$ coincides with the rotation on $\Delta\left((j+1)\cdot\varepsilon\right)$ and with the identity on $\mathbb{R}^2 \setminus \Delta\left(j\cdot\varepsilon\right)$. From this we conclude that $\det\left(D\bar{\varphi}_{\varepsilon}\right) > 0$. Moreover $\bar{\varphi}_{\varepsilon}$ is measure-preserving on $U_{\varepsilon} := \left(\mathbb{R}^2 \setminus \Delta\left(j\cdot\varepsilon\right)\right) \cup \Delta\left((j+1)\cdot\varepsilon\right)$.
As in the proof of [Kun13a], Lemma 2.4., we construct a diffeomorphism φ_{ε}, that is measure-preserving on the whole $\mathbb{R}^2$ and agrees with $\bar{\varphi}_{\varepsilon}$ on U_{ε} with the aid of "Moser's trick". □

Furthermore, for $\lambda \in \mathbb{N}$ we define the maps $C_{\lambda}\left(x_1, x_2\right) = \left(\lambda \cdot x_1, x_2\right)$ and $D_{\lambda}\left(x_1, x_2\right) = \left(\lambda \cdot x_1, \lambda \cdot x_2\right)$. Let $\mu \in \mathbb{N}$, $\frac{1}{\delta} \in \mathbb{N}$ and $\frac{1}{\delta}$ divides μ. We construct a diffeomorphism $\psi_{\mu,\delta,\epsilon_2}$ in the following way:

- Consider $[0, 1-2\delta]^2$: Since $\frac{1}{\delta}$ divides μ we can divide $[0, 1-2\delta]^2$ in cubes of sidelength $\frac{1}{\mu}$.
- Under the map D_{μ} any of these cubes of the form $\prod_{i=1}^{2}\left[\frac{l_i}{\mu}, \frac{l_i+1}{\mu}\right]$ with $l_i \in \mathbb{N}$ is mapped onto $\prod_{i=1}^{2}\left[l_i, l_i+1\right]$.
- On $[0,1]^2$ we will use the diffeomorphism $\varphi_{\epsilon_2,1}^{-1}$ constructed in Lemma 2.10. Since this is the identity outside of $\Delta\left(\epsilon_2\right)$ we can extend it to a diffeomorphism $\bar{\varphi}_{\epsilon_2,1}^{-1}$ on $\mathbb{R}^2$ using the instruction $\bar{\varphi}_{\epsilon_2,1}^{-1}\left(x_1+k_1, x_2+k_2\right) = \left(k_1, k_2\right) + \varphi_{\epsilon_2,1}^{-1}\left(x_1, x_2\right)$, where $k_i \in \mathbb{Z}$ and $x_i \in [0,1]$.
- Now we define the smooth measure-preserving diffeomorphism
$$\tilde{\psi}_{\mu,\delta,\epsilon_2} = D_{\mu}^{-1} \circ \bar{\varphi}_{\epsilon_2,1}^{-1} \circ D_{\mu} \quad : \quad [0, 1-2\delta]^2 \to [0, 1-2\delta]^2$$
- Hereby we define
$$\psi_{\mu,\delta,\epsilon_2}\left(x_1, x_2\right) = \begin{cases} \left(\left[\tilde{\psi}_{\mu,\delta,\epsilon_2}\left(x_1-\delta, x_2-\delta\right)\right]_1 + \delta, \left[\tilde{\psi}_{\mu,\delta,\epsilon_2}\left(x_1-\delta, x_2-\delta\right)\right]_2 + \delta\right) & \textit{on } [\delta, 1-\delta]^2 \\ \left(x_1, x_2\right) & \textit{else} \end{cases}$$
This is a smooth map because $\tilde{\psi}_{\mu,\delta,\epsilon_2}$ is the identity in a neighbourhood of the boundary by the construction.

Remark 2.11. For every set $W = \prod_{i=1}^{2}\left[\frac{l_i}{\mu} + r_i, \frac{l_i+1}{\mu} - r_i\right]$, where $l_i \in \mathbb{Z}$ and $r_i \in \mathbb{R}$ satisfies $|r_i \cdot \mu| \leq \epsilon_2$, we have $\psi_{\mu,\delta,\epsilon_2}(W) = W$.

Using these maps we build the following smooth measure-preserving diffeomorphism $\tilde{\phi}_{\lambda,\epsilon,j,\mu,\delta,\epsilon_2}$: $\left[0, \frac{1}{\lambda}\right] \times \mathbb{R} \to \left[0, \frac{1}{\lambda}\right] \times \mathbb{R}$:
$$\tilde{\phi}_{\lambda,\epsilon,j,\mu,\delta,\epsilon_2} = C_{\lambda}^{-1} \circ \psi_{\mu,\delta,\epsilon_2} \circ \varphi_{\epsilon,j} \circ C_{\lambda}$$
Afterwards $\tilde{\phi}_{\lambda,\epsilon,j,\mu,\delta,\epsilon_2}$ is extended to a diffeomorphism on $\mathbb{S}^1 \times \mathbb{R}$ by the description
$$\tilde{\phi}_{\lambda,\epsilon,j,\mu,\delta,\epsilon_2}\left(x_1 + \frac{k_1}{\lambda}, x_2 + k_2\right) = \left(\frac{k_1}{\lambda}, k_2\right) + \tilde{\phi}_{\lambda,\epsilon,j,\mu,\delta,\epsilon_2}\left(x_1, x_2\right)$$
for $k_i \in \mathbb{Z}$.

For convenience we will use the following notation: $\tilde{\phi}_{\lambda,\mu} = \tilde{\phi}_{\lambda,\varepsilon_n,4n^2,\mu,4n^2\cdot\varepsilon_n,\frac{\varepsilon_n}{3}}$. Hereby we define the diffeomorphism ϕ_n on the fundamental sector: On $\left[\frac{k}{n\cdot q_n}, \frac{k+1}{n\cdot q_n}\right] \times \mathbb{R}$ in case of $k \in \mathbb{N}$ and $0 \leq k \leq n-1$
$$\phi_n = \tilde{\phi}_{n\cdot q_n^{2+3+4+\ldots+(3+k-1)}, q_n^{3+k}} = \tilde{\phi}_{n\cdot q_n^{2+3k+\frac{k\cdot(k-1)}{2}}, q_n^{3+k}}$$
Now we extend ϕ_n to a diffeomorphism on $\mathbb{S}^1 \times \mathbb{R}$ using the description $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$.

Remark 2.12. Since $\varphi_{\varepsilon,j}$ coincides with the identity outside of $\Delta(j\cdot\varepsilon)=[j\cdot\varepsilon,1-j\cdot\varepsilon]^2$ we have $\phi_n\left(D_{\psi_n,\gamma_n}\left(\mathbb{S}^1\times[0,1]\right)\right)=D_{\psi_n,\gamma_n}\left(\mathbb{S}^1\times[0,1]\right)$. Hence $D^{-1}_{\psi_n,\gamma_n}\circ\phi_n\circ D_{\psi_n,\gamma_n}:\mathbb{S}^1\times[0,1]\to\mathbb{S}^1\times[0,1]$.

3 Convergence of $(f_n)_{n\in\mathbb{N}}$ in Diff$^\infty\left(\mathbb{S}^1\times[0,1],\mu\right)$

In the following we show that the sequence of constructed measure-preserving smooth diffeomorphisms $f_n=H_n\circ R_{\alpha_{n+1}}\circ H_n^{-1}$ converges. For this purpose we need a couple of results concerning the conjugation maps.

3.1 Properties of the conjugation maps ϕ_n and H_n

In this subsection we want to find estimates on the norms $|||H_n|||_k$. For this we have to estimate the norms of the occurrent maps.

Lemma 3.1. *For every $k\in\mathbb{N}$ it holds*

$$|||D_{\psi_n,\gamma_n}|||_{k,ext}\leq C\cdot\gamma_n^k$$

where C is a constant depending on n and k, but is independent of q_n.

Proof. By construction of the map $D_{\psi_n,\gamma_n}=C^{-1}_{\gamma_n}\circ\bar{D}_{\psi_n}\circ C_{\gamma_n}$ we have

$$D_{\psi_n,\gamma_n}(\theta,r)=(\theta,r+d_n\cdot\psi_n(\gamma_n\cdot\theta))$$

as well as

$$D^{-1}_{\psi_n,\gamma_n}(\theta,r)=(\theta,r-d_n\cdot\psi_n(\gamma_n\cdot\theta))$$

using the abbreviation $d_n:=1+\frac{1}{q_n}+...+\frac{1}{q_n^{3+n-1}}$.

Since $d_n\leq 2$ we obtain: $|||D_{\psi_n,\gamma_n}|||_{k,\text{ext}}\leq\tilde{C}\cdot d_n\cdot\gamma_n^k\leq C\cdot q_n^{k\cdot\left(2+3\cdot n+\frac{n\cdot(n-1)}{2}\right)}$. □

Remark 3.2. In the proof of the following Lemmas we will use the formula of Faà di Bruno in several variables. It can be found in the paper *"A multivariate Faà di Bruno formula with applications"* ([CS96]) for example.

Therefor we introduce an ordering on $\mathbb{N}_0^d$: For multiindices $\vec{\mu}=(\mu_1,...,\mu_d)$ and $\vec{\nu}=(\nu_1,...,\nu_d)$ in $\mathbb{N}_0^d$ we will write $\vec{\mu}\prec\vec{\nu}$, if one of the following properties is satisfied:

1. $|\vec{\mu}|<|\vec{\nu}|$, where $|\vec{\mu}|=\sum_{i=1}^d\mu_i$.
2. $|\vec{\mu}|=|\vec{\nu}|$ and $\mu_1<\nu_1$
3. $|\vec{\mu}|=|\vec{\nu}|$, $\mu_i=\nu_i$ for $1\leq i\leq k$ and $\mu_{k+1}<\nu_{k+1}$ for a $1\leq k<d$

Additionally we will use these notations:

- For $\vec{\nu}=(\nu_1,...,\nu_d)\in\mathbb{N}_0^d$:

$$\vec{\nu}!=\prod_{i=1}^d\nu_i!$$

- For $\vec{\nu} = (\nu_1, ..., \nu_d) \in \mathbb{N}_0^d$ and $\vec{z} = (z_1, ..., z_d) \in \mathbb{R}^d$:

$$\vec{z}^{\,\vec{\nu}} = \prod_{i=1}^{d} z_i^{\nu_i}$$

Then we get for the composition $h(x_1, ..., x_d) := f\left(g^{(1)}(x_1, ..., x_d), ..., g^{(m)}(x_1, ..., x_d)\right)$ with sufficiently differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$, $g^{(i)} : \mathbb{R}^d \to \mathbb{R}$ and a multiindex $\vec{\nu} \in \mathbb{N}_0^d$ with $|\vec{\nu}| = n$:

$$D_{\vec{\nu}} h = \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq n} D_{\vec{\lambda}} f \cdot \sum_{s=1}^{n} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\left[D_{\vec{l}_j}\vec{g}\right]^{\vec{k}_j}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

Hereby $\left[D_{\vec{l}_j}\vec{g}\right]$ denotes $\left(D_{\vec{l}_j} g^{(1)}, ..., D_{\vec{l}_j} g^{(m)}\right)$ and

$p_s\left(\vec{\nu}, \vec{\lambda}\right) :=$

$$\left\{\left(\vec{k}_1, ..., \vec{k}_s, \vec{l}_1, ..., \vec{l}_s\right) : \vec{k}_i \in \mathbb{N}_0^m, \left|\vec{k}_i\right| > 0, \vec{l}_i \in \mathbb{N}_0^d, 0 \prec \vec{l}_1 \prec ... \prec \vec{l}_s, \sum_{i=1}^{s} \vec{k}_i = \vec{\lambda} \text{ and } \sum_{i=1}^{s} \left|\vec{k}_i\right| \cdot \vec{l}_i = \vec{\nu}\right\}$$

With the aid of these technical results we can prove an estimate on the norms of the map ϕ_n:

Lemma 3.3. *For every $k \in \mathbb{N}$ it holds*

$$|||\phi_n|||_{k,ext} \leq C \cdot \gamma_n^k$$

where C is a constant depending on k and n, but is independent of q_n.

Proof. First of all we consider the map $\tilde{\phi}_{\lambda,\mu} := \tilde{\phi}_{\lambda,\epsilon,j,\mu,\delta,\epsilon_2} = C_{\lambda}^{-1} \circ \psi_{\mu,\delta,\epsilon_2} \circ \varphi_{\epsilon,j} \circ C_{\lambda}$ introduced in subsection 2.5:

$$\tilde{\phi}_{\lambda,\mu}(x_1, x_2) = \left(\frac{1}{\lambda}\left[\psi_\mu \circ \varphi_{\epsilon,j}\right]_1(\lambda x_1, x_2), \left[\psi_\mu \circ \varphi_{\epsilon,j}\right]_2(\lambda x_1, x_2)\right)$$

Let $k \in \mathbb{N}$. We compute for a multiindex $\vec{a}$ with $0 \leq |\vec{a}| \leq k$:

$$\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}\right]_1\right\|_{0,\text{ext}} \leq \lambda^{k-1} \cdot |||\psi_\mu \circ \varphi_{\epsilon,j}|||_{k,\text{ext}} \text{ and } \left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}\right]_2\right\|_{0,\text{ext}} \leq \lambda^{k} \cdot |||\psi_\mu \circ \varphi_{\epsilon,j}|||_{k,\text{ext}}.$$

Therefore we examine the map ψ_μ. For any multiindex $\vec{a}$ with $0 \leq |\vec{a}| \leq k$ and $u \in \{1,2\}$ we obtain: $\left\|D_{\vec{a}}\left[\psi_\mu\right]_u\right\|_{0,\text{ext}} \leq \mu^{k-1} \cdot |||\varphi_{\epsilon_2}|||_{k,\text{ext}} = C_{k,\epsilon_2} \cdot \mu^{k-1}$ and in the same way we get $\left\|D_{\vec{a}}\left[\psi_\mu^{-1}\right]_u\right\|_{0,\text{ext}} \leq C_{k,\epsilon_2} \cdot \mu^{k-1}$. Hence: $|||\psi_\mu|||_{k,\text{ext}} \leq C \cdot \mu^{k-1}$.
In the next step we use the formula of Faà di Bruno mentioned in remark 3.2. With it we compute for any multiindex $\vec{\nu}$ with $|\vec{\nu}| = k$:

$$\begin{aligned}\left\|D_{\vec{\nu}}\left[\left(\psi_\mu \circ \varphi_{\epsilon,j}\right)^{-1}\right]_u\right\|_{0,\text{ext}} &= \left\|D_{\vec{\nu}}\left[\varphi_{\epsilon,j}^{-1} \circ \psi_\mu^{-1}\right]_u\right\|_{0,\text{ext}} \\ &= \left\|\sum_{\vec{\lambda}\in\mathbb{N}_0^2, 1\leq|\vec{\lambda}|\leq k} D_{\vec{\lambda}}\left[\varphi_{\epsilon,j}^{-1}\right]_u \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{i=1}^{s} \frac{\left[D_{\vec{l}_i}\psi_\mu^{-1}\right]^{\vec{k}_i}}{\vec{k}_i! \cdot \left(\vec{l}_i!\right)^{|\vec{k}_i|}}\right\|_{0,\text{ext}}\end{aligned}$$

$$\begin{aligned}
&= \left\| \sum_{\vec{\lambda}\in\mathbb{N}_0^2, 1\leq|\vec{\lambda}|\leq k} D_{\vec{\lambda}}\left[\varphi_{\epsilon,j}^{-1}\right]_u \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{i=1}^{s} \frac{\prod_{t=1}^{m}\left(D_{\vec{l}_i}\left[\psi_\mu^{-1}\right]_t\right)^{\vec{k}_{i_t}}}{\vec{k}_i!\cdot\left(\vec{l}_i!\right)^{|\vec{k}_i|}} \right\|_{0,\text{ext}} \\
&\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^2, 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\varphi_{\epsilon,j}^{-1}\right]_u\right\|_{0,\text{ext}} \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{i=1}^{s} \frac{\prod_{t=1}^{m}\left\|D_{\vec{l}_i}\left[\psi_\mu^{-1}\right]_t\right\|_{0,\text{ext}}^{\vec{k}_{i_t}}}{\vec{k}_i!\cdot\left(\vec{l}_i!\right)^{|\vec{k}_i|}} \\
&\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^2 \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\varphi_{\epsilon,j}^{-1}\right]_u\right\|_{0,\text{ext}} \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{i=1}^{s} \frac{|||\psi_\mu^{-1}|||_{|\vec{l}_i|,\text{ext}}^{\sum_{t=1}^{m}\vec{k}_{i_t}}}{\vec{k}_i!\cdot\left(\vec{l}_i!\right)^{|\vec{k}_i|}} \\
&= \sum_{\vec{\lambda}\in\mathbb{N}_0^2 \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\varphi_{\epsilon,j}^{-1}\right]_u\right\|_{0,\text{ext}} \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{i=1}^{s} \frac{|||\psi_\mu^{-1}|||_{|\vec{l}_i|,\text{ext}}^{|\vec{k}_i|}}{\vec{k}_i!\cdot\left(\vec{l}_i!\right)^{|\vec{k}_i|}}
\end{aligned}$$

As seen above: $|||\psi_\mu^{-1}|||_{|\vec{l}_i|,\text{ext}}^{|\vec{k}_i|} \leq C\cdot\mu^{|\vec{k}_i|\cdot|\vec{l}_i|}$. Hereby: $\prod_{i=1}^{s}|||\psi_\mu^{-1}|||_{|\vec{l}_i|,\text{ext}}^{|\vec{k}_i|} \leq \hat{C}\cdot\mu^{\sum_{i=1}^{s}|\vec{l}_i|\cdot|\vec{k}_i|}$ where $\hat{C}$ is independent of μ. By definition of the set $p_s\left(\vec{\nu},\vec{\lambda}\right)$ we have $\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i=\vec{\nu}$. Hence:

$$k=|\vec{\nu}|=\left|\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i\right|=\sum_{t=1}^{2}\left(\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i\right)_t=\sum_{t=1}^{2}\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_{i_t}=\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\left(\sum_{t=1}^{2}\vec{l}_{i_t}\right)=\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\left|\vec{l}_i\right|$$

This shows $\prod_{i=1}^{s}|||\psi_\mu^{-1}|||_{|\vec{l}_i|,\text{ext}}^{|\vec{k}_i|}\leq\hat{C}\cdot\mu^k$ and finally $\left\|D_{\vec{\nu}}\left[\left(\psi_\mu\circ\varphi_{\epsilon,j}\right)^{-1}\right]_u\right\|_{0,\text{ext}}\leq C\cdot\mu^k$. Analogously we compute $\left\|D_{\vec{\nu}}\left[\psi_\mu\circ\varphi_{\epsilon,j}\right]_u\right\|_{0,\text{ext}}\leq C\cdot|||\psi_\mu|||_{k,\text{ext}}\leq C\cdot\mu^{k-1}$. Altogether we obtain $|||\psi_\mu\circ\varphi_{\epsilon,j}|||_{k,\text{ext}}\leq C\cdot\mu^k$. Hereby we estimate $\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}\right]_u\right\|_{0,\text{ext}}\leq C\cdot\lambda^k\cdot\mu^k$ and analogously $\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}^{-1}\right]_u\right\|_{0,\text{ext}}\leq C\cdot\lambda^k\cdot\mu^k$. In conclusion this yields $|||\tilde{\phi}_{\lambda,\mu}|||_{k,\text{ext}}\leq C\cdot\mu^k\cdot\lambda^k$.
In the setting of our explicit construction of the map ϕ_n in section 2.5 we have $\epsilon=\varepsilon_n$, $\epsilon_2=\frac{\varepsilon_n}{3}$, $\lambda_{max}=n\cdot q_n^{2+3\cdot(n-1)+\frac{(n-1)\cdot(n-2)}{2}}$ and $\mu_{max}=q_n^{3+n-1}$. Thus:

$$\begin{aligned}
|||\phi_n|||_{k,\text{ext}} &\leq \tilde{C}(k,n)\cdot\left(n\cdot q_n^{2+3\cdot(n-1)+\frac{(n-1)\cdot(n-2)}{2}}\right)^k\cdot\left(q_n^{3+n-1}\right)^k \\
&\leq C(k,n)\cdot\gamma_n^k
\end{aligned}$$

where $C(k,n)$ is a constant independent of q_n. □

Combining the last results with the aid of the formula of Faà di Bruno yields

Lemma 3.4. *For every $k\in\mathbb{N}$ we have:*

$$\left|\left|\left|D_{\psi_n,\gamma_n}^{-1}\circ\phi_n\circ D_{\psi_n,\gamma_n}\right|\right|\right|_k\leq C\cdot\gamma_n^{3\cdot k}$$

where C is a constant depending on k and n, but is independent of q_n.

In the next step we consider the map $h_n=g_n\circ D_{\psi_n,\gamma_n}^{-1}\circ\phi_n\circ D_{\psi_n,\gamma_n}$, where g_n is constructed in section 2.4:

Lemma 3.5. *For every $k \in \mathbb{N}$ we have:*

$$|||h_n|||_k \le C\cdot q_n^k\cdot\gamma_n^{4\cdot k}$$

where C is a constant depending on k and n, but is independent of q_n.

Proof. We label $\bar{\phi}_n := D^{-1}_{\psi_n,\gamma_n}\circ\phi_n\circ D_{\psi_n,\gamma_n}$. Outside of $\mathbb{S}^1\times[\delta,1-\delta]^{m-1}$ we have:

$$\begin{aligned} h_n(x_1,x_2) &= g_n\circ\bar{\phi}_n(x_1,x_2)\\ &= \left(\left[\bar{\phi}_n(x_1,x_2)\right]_1+\chi_n(x_2)\cdot\left[n\cdot q_n^{\sigma_n}\right]\cdot\left[\bar{\phi}_n(x_1,x_2)\right]_2,\left[\bar{\phi}_n(x_1,x_2)\right]_2\right)\end{aligned}$$

and

$$\begin{aligned} h_n^{-1}(x_1,x_2) &= \bar{\phi}_n^{-1}\circ g_n^{-1}(x_1,x_2)\\ &= \left(\left[\bar{\phi}_n^{-1}\left(x_1-\chi_n(x_2)\cdot\left[n\cdot q_n^{\sigma_n}\right]\cdot x_2,x_2\right)\right]_1,\left[\bar{\phi}_n\left(x_1-\chi_n(x_2)\cdot\left[n\cdot q_n^{\sigma_n}\right]\cdot x_2,x_2\right)\right]_2\right)\end{aligned}$$

Since $\sigma_n<1$ we can estimate:

$$|||h_n|||_k \le 2\cdot C_{n,k}\cdot\left[n\cdot q_n^{\sigma_n}\right]^k\cdot|||\bar{\phi}_n|||_k \le \bar{C}\cdot q_n^{\sigma_n\cdot k}\cdot\gamma_n^{3\cdot k}\le\bar{C}\cdot q_n^k\cdot\gamma_n^{3\cdot k}$$

with a constant $\bar{C}$ independent of q_n.

In the other case we have

$$g_n\circ\bar{\phi}_n(x_1,x_2)=\left(\left[g_{a,b,\varepsilon}\left(\left[\bar{\phi}_n\right]_1,\left[\bar{\phi}_n\right]_2\right)\right]_1(x_1,x_2),\left[g_{a,b,\varepsilon}\left(\left[\bar{\phi}_n\right]_1,\left[\bar{\phi}_n\right]_2\right)\right]_2(x_1,x_2)\right)$$

We will use the formula of Faà di Bruno as above for any multiindex $\vec{\nu}$ with $|\vec{\nu}|=k$ and obtain for $u\in\{1,2\}$:

$$\begin{aligned}\left\|D_{\vec{\nu}}\left[g_n\circ\bar{\phi}_n\right]_u\right\|_0 &= \left\|D_{\vec{\nu}}\left[g_{a,b,\varepsilon}\circ\bar{\phi}_n\right]_u\right\|_0\\ &\le \sum_{\vec{\lambda}\in\mathbb{N}_0^2\text{ with }1\le|\vec{\lambda}|\le k}\left\|D_{\vec{\lambda}}\left[g_{a,b,\varepsilon}\right]_u\right\|_0\cdot\sum_{s=1}^{k}\sum_{p_s(\vec{\nu},\vec{\lambda})}\vec{\nu}!\cdot\prod_{j=1}^{s}\frac{|||\bar{\phi}_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}\end{aligned}$$

By Lemma 3.4 we have $|||\bar{\phi}_n|||_k\le C\cdot\gamma_n^{3\cdot k}$, where C is a constant independent of q_n. As above we show $\prod_{j=1}^{s}|||\bar{\phi}_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}\le\hat{C}\cdot\gamma_n^{\left(\sum_{j=1}^{s}|\vec{l}_j|\cdot|\vec{k}_j|\right)\cdot 3}=\hat{C}\cdot\gamma_n^{k\cdot 3}$, where $\hat{C}$ is a constant independent of q_n.

Furthermore we examine the map $g_{a,b,\varepsilon}$ for $a,b\in\mathbb{Z}$:

$$\begin{aligned} g_{a,b,\varepsilon}(x_1,x_2) &= \left(\frac{1}{a}\cdot[g_\varepsilon]_1\left(a\cdot x_1,\frac{b\cdot a}{\varepsilon}\cdot x_2\right),\frac{\varepsilon}{b\cdot a}[g_\varepsilon]_2\left(a\cdot x_1,\frac{b\cdot a}{\varepsilon}\cdot x_2\right)\right)\\ g^{-1}_{a,b,\varepsilon}(x_1,x_2) &= \left(\frac{1}{a}\cdot\left[g_\varepsilon^{-1}\right]_1\left(a\cdot x_1,\frac{b\cdot a}{\varepsilon}\cdot x_2\right),\frac{\varepsilon}{b\cdot a}\left[g_\varepsilon^{-1}\right]_2\left(a\cdot x_1,\frac{b\cdot a}{\varepsilon}\cdot x_2\right)\right)\end{aligned}$$

Thus: $|||g_{a,b,\varepsilon}|||_k\le\left(\frac{b\cdot a}{\varepsilon}\right)^{k-1}\cdot\frac{b}{\varepsilon}\cdot|||g_\varepsilon|||_k=C_{\varepsilon,k}\cdot b^k\cdot a^{k-1}$. By our constructions in section 2.4 we have $b=\left[n\cdot q_n^{\sigma_n}\right]\le n\cdot q_n^{\sigma_n}$, $a\le\gamma_n$ and $\varepsilon=16n^2\cdot\varepsilon_n$. Hence:

$$|||g_n|||_k\le C_{n,k}\cdot q_n^{\sigma_n\cdot k}\cdot\gamma_n^{k-1}\le C_{n,k}\cdot q_n^k\cdot\gamma_n^{k-1}.$$

Finally we conclude: $\left\|D_{\vec{\nu}}\left[g_n\circ\bar{\phi}_n\right]_u\right\|_0 \leq C(n,k)\cdot q_n^k\cdot\gamma_n^{k-1}\cdot\gamma_n^{3\cdot k}\leq C(n,k)\cdot q_n^k\cdot\gamma_n^{4\cdot k}$.
In the next step we consider the inverse $\bar{\phi}_n^{-1}\circ g_n^{-1}$:

$$\begin{aligned}&\bar{\phi}_n^{-1}\circ g_{a,b,\varepsilon}^{-1}(x_1,x_2)=\\&\left(\left[\bar{\phi}_n^{-1}\right]_1\left(\left[g_{a,b,\varepsilon}^{-1}\right]_1(x_1,x_2),\left[g_{a,b,\varepsilon}^{-1}\right]_1(x_1,x_2)\right),\left[\phi_n^{-1}\right]_2\left(\left[g_{a,b,\varepsilon}^{-1}\right]_1(x_1,x_2),\left[g_{a,b,\varepsilon}^{-1}\right]_1(x_1,x_2)\right)\right)\end{aligned}$$

For $u\in\{1,2\}$ and any multiindex $\vec{\nu}$ with $|\vec{\nu}|=k$ we obtain using the formula of Faà di Bruno again:

$$\left\|D_{\vec{\nu}}\left[\bar{\phi}_n^{-1}\circ g_n^{-1}\right]_u\right\|_0\leq\sum_{\vec{\lambda}\in\mathbb{N}_0^2\text{ with }1\leq|\vec{\lambda}|\leq k}\left\|D_{\vec{\lambda}}\left[\bar{\phi}_n^{-1}\right]_u\right\|_0\cdot\sum_{s=1}^{k}\sum_{p_s(\vec{\nu},\vec{\lambda})}\vec{\nu}!\cdot\prod_{j=1}^{s}\frac{|||g_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

As above we show $\prod_{j=1}^{s}|||g_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}\leq\hat{C}\cdot q_n^k\cdot\gamma_n^k$, where $\hat{C}$ is a constant independent of q_n. Since $|||\bar{\phi}_n|||_k\leq C\cdot\gamma_n^{3\cdot k}$ we get

$$\left\|D_{\vec{\nu}}\left[\bar{\phi}_n^{-1}\circ g_n^{-1}\right]_u\right\|_0\leq\check{C}\cdot q_n^k\cdot\gamma_n^k\cdot\gamma_n^{3\cdot k}\leq\check{C}\cdot q_n^k\cdot\gamma_n^{4\cdot k}$$

where $\check{C}$ is a constant independent of q_n.
Thus we obtain finally $|||g_n\circ\bar{\phi}_n|||_k\leq C(n,k)\cdot q_n^k\cdot\gamma_n^{4\cdot k}$. □

Finally we are able to prove an estimate on the norms of the map H_n:

Lemma 3.6. *For every $k\in\mathbb{N}$ we get:*

$$|||H_n|||_k\leq\check{C}\cdot q_n^{k\cdot 4\cdot n\cdot(n+5)}$$

where $\check{C}$ is a constant depending solely on k, n and H_{n-1}. Since H_{n-1} is independent of q_n in particular, the same is true for $\check{C}$.

Proof. By Lemma 3.5 and $\gamma_n=n\cdot q_n^{2+3n+\frac{(n-1)\cdot n}{2}}=n\cdot q_n^{2+\frac{n\cdot(n+5)}{2}}$ we have

$$|||h_n|||_k\leq C\cdot q_n^{k\cdot\left(1+4\cdot 2+4\cdot\frac{n\cdot(n+5)}{2}\right)}\leq C\cdot q_n^{k\cdot 4\cdot n\cdot(n+5)}$$

Let $k\in\mathbb{N}$, $u\in\{1,2\}$ and $\vec{\nu}\in\mathbb{N}_0^2$ be a multiindex with $|\vec{\nu}|=k$. As above we estimate:

$$\begin{aligned}\left\|D_{\vec{\nu}}\left[H_n\right]_u\right\|_0&=\left\|D_{\vec{\nu}}\left[H_{n-1}\circ h_n\right]_u\right\|_0\\&\leq\sum_{\vec{\lambda}\in\mathbb{N}_0^2\text{ with }1\leq|\vec{\lambda}|\leq k}\left\|D_{\vec{\lambda}}\left[H_{n-1}\right]_u\right\|_0\cdot\sum_{s=1}^{k}\sum_{p_s(\vec{\nu},\vec{\lambda})}\vec{\nu}!\cdot\prod_{j=1}^{s}\frac{|||h_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}\end{aligned}$$

and compute using Lemma 3.5: $\prod_{j=1}^{s}|||h_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}\leq\hat{C}\cdot q_n^{k\cdot 4\cdot n\cdot(n+5)}$. Since H_{n-1} was constructed independently of q_n we conclude: $\left\|D_{\vec{\nu}}\left[H_n\right]_u\right\|_0\leq\check{C}\cdot q_n^{k\cdot 4\cdot n\cdot(n+5)}$, where $\check{C}$ is a constant independent of q_n.
In the same way we prove an analogous estimate on $\left\|D_{\vec{\nu}}\left[H_n^{-1}\right]_u\right\|_0$ and verify the claim. □

3.2 Proof of convergence

In [Kun13a], Lemma 5.8., we proved that under some assumptions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{A}_\alpha$ in the $\mathrm{Diff}^{\infty}(M)$-topology.

Lemma 3.7. *Let $\varepsilon > 0$ be arbitrary and $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^{\infty} \frac{1}{k_n} < \epsilon$. For each $k_n \in \mathbb{N}$ there is a constant $C_{k_n} \geq 1$ determined by [Kun13a], Lemma 5.7.. Furthermore we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha - \alpha_1| < \epsilon \qquad \text{and} \qquad |\alpha - \alpha_n| \leq \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}} \text{ for every } n \in \mathbb{N}.$$

1. *Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the $\mathrm{Diff}^{\infty}(M)$-topology to a measure-preserving smooth diffeomorphism f, for which $d_{\infty}(f, R_\alpha) < 3 \cdot \epsilon$ holds.*
2. *Also the sequence of diffeomorphisms $\hat{f}_n = H_n \circ R_\alpha \circ H_n^{-1} \in \mathcal{A}_\alpha(M)$ converges to f in the $\mathrm{Diff}^{\infty}(M)$-topology. Hence $f \in \mathcal{A}_\alpha$.*

Next we show that we can satisfy the conditions from Lemma 3.7 in our constructions:

Lemma 3.8. *Let $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing seq. of natural numbers with $\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty$ and C_{k_n} be the constants from Lemma 3.7. For any Liouvillean number α there exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers with $\frac{1}{\varepsilon_n}$ divides q_n and $q_n > \max_{i=1,\dots,n+1} L_i$ (where L_i denotes the Lipschitz constant of $\rho_i \in \Xi$), such that our conjugation maps H_n constructed in section 2 fulfil the following conditions:*

1. *For every $n \in \mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}}$$
2. *For every $n \in \mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot q_n \cdot |||H_n|||_1}$$

Proof. In Lemma 3.6 we deduced the estimate $|||H_n|||_{k_n+1} \leq \breve{C}_n \cdot q_n^{(k_n+1)\cdot 4 \cdot n \cdot (n+5)}$, where the constant $\breve{C}_n$ was independent of q_n. Thus we can choose $q_n \geq \breve{C}_n$ for every $n \in \mathbb{N}$. Hence we obtain: $|||H_n|||_{k_n+1} \leq q_n^{8\cdot n\cdot(n+5)\cdot(k_n+1)}$. Besides $q_n \geq \breve{C}_n$ we set the conditions $q_n > n^{13} \cdot q_{n-1}^{9\cdot r(n-1)+1}$ and $q_n \geq \frac{1}{\varepsilon_n} \cdot 12 \cdot \frac{1}{\varepsilon_{n-1}} \cdot |||\psi_{n-1}|||_1 \cdot q_{n-1}^{2\cdot\left(2+3\cdot(n-1)+\frac{(n-1)\cdot(n-2)}{2}\right)}$. Furthermore we can demand $q_n > \max_{i=1,\dots,n+1} L_i$. Since α is a Liouvillean number we find a sequence of rational numbers $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$, $\tilde{p}_n, \tilde{q}_n$ relatively prime, converging to α under the above restrictions (formulated for $\tilde{q}_n$) satisfying:

$$|\alpha - \tilde{\alpha}_n| = \left|\alpha - \frac{\tilde{p}_n}{\tilde{q}_n}\right| < \frac{|\alpha - \alpha_{n-1}| \cdot \varepsilon_n^{1+8\cdot n\cdot(n+5)\cdot(k_n+1)^2}}{2^{n+1} \cdot k_n \cdot C_{k_n} \cdot \tilde{q}_n^{1+8\cdot n\cdot(n+5)\cdot(k_n+1)^2}}$$

Put $q_n := \frac{\tilde{q}_n}{\varepsilon_n}$ and $p_n := \frac{\tilde{p}_n}{\varepsilon_n}$. Then we obtain:

$$|\alpha - \alpha_n| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1} \cdot k_n \cdot C_{k_n} \cdot q_n^{1+8\cdot n\cdot(n+5)\cdot(k_n+1)^2}}.$$

Thus we have $|\alpha-\alpha_n| \overset{n\to\infty}{\to} 0$ monotonically. Because of $|||H_n|||_{k_n+1}^{k_n+1} \leq q_n^{8\cdot n\cdot(n+5)\cdot(k_n+1)^2}$ this yields: $|\alpha-\alpha_n| < \frac{1}{2^{n+1}\cdot q_n\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}}$. Thus the first property of this Lemma is fulfilled. Furthermore we note $k_n \geq 1$ and $C_{k_n} \geq 1$ by the assumption in Lemma 3.7. Thus $q_n\cdot k_n\cdot C_{k_n} \geq q_n$. Moreover $|||H_n|||_1 \geq \|H_n\|_0 = 1$, because $H_n : \mathbb{S}^1\times[0,1]^{m-1} \to \mathbb{S}^1\times[0,1]^{m-1}$ is a diffeomorphism. Altogether we conclude $2^{n+1}\cdot q_n\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1} \geq 2^{n+1}\cdot q_n\cdot|||H_n|||_1$ and so:

$$|\alpha-\alpha_n| < \frac{1}{2^{n+1}\cdot q_n\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}} \leq \frac{1}{2^{n+1}\cdot q_n\cdot|||H_n|||_1}. \tag{1}$$

i.e. we verified the second property. □

Remark 3.9. Lemma 3.8 shows that the conditions of Lemma 3.7 are satisfied. Therefore our sequence of constructed diffeomorphisms f_n converges in the $\mathrm{Diff}^\infty(M)$-topology to a diffeomorphism $f \in \mathcal{A}_\alpha(M)$.

Remark 3.10. In particular $|||H_n|||_1 \leq q_n^{8\cdot n\cdot(n+5)}$ motivates our definition of the number $r(n) = 8\cdot n\cdot(n+5)$.

As in [Kun13a], Lemma 5.11., we can conclude:

Lemma 3.11. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be constructed as in Lemma 3.8. Then it holds for every $n\in\mathbb{N}$ and for every $\tilde{m} \leq q_{n+1}$:*

$$d_0\left(f^{\tilde{m}}, f_n^{\tilde{m}}\right) \leq \frac{1}{2^n}$$

Remark 3.12. We determine the parameter $\sigma_n \in (0,1)$ in such a way that $q_n^{\sigma_n} = q_{n-1}^{4\cdot r(n-1)}$, i.e. we have $[nq_n^{\sigma_n}] = n\cdot q_{n-1}^{4\cdot r(n-1)}$.

4 The invariant measures

As above μ is the Lebesgue measure on $\mathbb{S}^1\times[0,1]$ and δ^0 (resp. δ^1) denotes the Lebesgue measure on the boundary component $\mathbb{S}^1\times\{0\}$ (resp. $\mathbb{S}^1\times\{1\}$). We aim for showing that these are the only ergodic f-invariant measures. Therefore we deduce a statement on the Birkhoff sums for arbitrary $x\in\mathbb{S}^1\times[0,1]$ (see Lemma 4.3). In order to prove such a statement we have to gain control over a large proportion of every R_t-orbit. This is done with the aid of the trapping maps and regions.
Furthermore $\tilde{\lambda}$ denotes the Lebesgue measure on $\mathbb{S}^1$ and λ the Lebesgue measure on $[0,1]$.

4.1 Trapping property

In case of $0\leq l\leq q_n-1$, $0\leq k\leq n-1$, $j_i^{(1)}\in\mathbb{Z}$, $\lceil 18n^2\varepsilon_n\cdot q_n\rceil \leq j_i^{(1)} \leq q_n - \lceil 18n^2\varepsilon_n\cdot q_n\rceil - 1$ for $i=1,2$ as well as $j_2^{(t)}\in\mathbb{Z}$, $0\leq j_2^{(t)}\leq q_n-1$ for $t=2,3$ we introduce the sets

$$\begin{aligned}&\Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}\\ &= \left[\frac{l}{q_n}+\frac{k}{n\cdot q_n}+\frac{j_1^{(1)}}{n\cdot q_n^2}, \frac{l}{q_n}+\frac{k}{n\cdot q_n}+\frac{j_1^{(1)}+1}{n\cdot q_n^2}\right]\times\left[\frac{j_2^{(1)}}{q_n}+\frac{j_2^{(2)}}{q_n^2}+\frac{j_2^{(3)}}{q_n^3}, \frac{j_2^{(1)}}{q_n}+\frac{j_2^{(2)}}{q_n^2}+\frac{j_2^{(3)}+1}{q_n^3}\right]\end{aligned}$$

Note that there are $q_n \cdot n \cdot \left(q_n - 2 \cdot \lceil 18n^2 \varepsilon_n \cdot q_n \rceil\right)^2 \cdot q_n^2$ such sets $\Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}$. We denote the union of these sets by T_n^{int} and the collection of these sets by $\tilde{T}_n^{\text{int}}$. Then

$$\mu\left(\mathbb{S}^1 \times [0,1] \setminus T_n^{\text{int}}\right) = 1 - n \cdot q_n \cdot \left(q_n - 2 \cdot \lceil 18n^2 \varepsilon_n \cdot q_n \rceil\right)^2 \cdot q_n^2 \cdot \frac{1}{n \cdot q_n^5} \leq 1 - \left(1 - 2 \cdot \frac{1}{4n^4}\right)^2 \leq \frac{1}{n^4}.$$

Note that $D_{\psi_n,\gamma_n}^{-1}\left(\Delta_{l,k,j_1^{(1)},\dots,j_2^{(3)}}\right) \subseteq \mathbb{S}^1 \times \left[12n^2 \cdot \varepsilon_n, 1 - 12n^2 \cdot \varepsilon_n\right]$. Unfortunately $g_n = \tilde{g}_{[nq_n^{\sigma_n}]}$ is not necessarily true on $D_{\psi_n,\gamma_n}^{-1}\left(\Delta_{l,k,j_1^{(1)},\dots,j_2^{(3)}}\right)$, but this set is strictly contained in a cube of sidelength $\frac{1}{n \cdot q_n^2} + 4n^2 \cdot \varepsilon_n \leq 8n^2 \cdot \varepsilon_n$ that is an union of domains of $g_{a,b.\varepsilon}$. Then we obtain

$$\begin{aligned}
\operatorname{diam}\left(H_{n-1} \circ g_n\left(D_{\psi_n,\gamma_n}^{-1}\left(\Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}\right)\right)\right) &\leq \|DH_{n-1}\|_0 \cdot n \cdot q_n^{\sigma_n} \cdot \sqrt{2} \cdot 8n^2 \cdot \varepsilon_n \\
&\leq q_{n-1}^{r(n-1)} \cdot q_{n-1}^{4 \cdot r(n-1)} \cdot 8n^3 \cdot \frac{\sqrt{2}}{4 \cdot n^{11} \cdot q_{n-1}^{5 \cdot r(n-1)+1}} \\
&< \frac{4}{n^8 \cdot q_{n-1}}
\end{aligned}$$

by the construction of the number σ_n in Remark 3.12.
By the requirements on the number q_n in Lemma 3.8 we obtain

$$\begin{aligned}
&\left|\rho_i\left(H_{n-1} \circ g_n \circ D_{\psi_n,\gamma_n}^{-1}(x)\right) - \rho_i\left(H_{n-1} \circ g_n \circ D_{\psi_n,\gamma_n}^{-1}(y)\right)\right| \\
\leq &\operatorname{Lip}(\rho_i) \cdot \operatorname{diam}\left(H_{n-1} \circ g_n\left(D_{\psi_n,\gamma_n}^{-1}\left(\Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}\right)\right)\right) \\
\leq &q_{n-1} \cdot \frac{4}{n^8 \cdot q_{n-1}} = \frac{4}{n^8}
\end{aligned}$$

for every $x, y \in \Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}$ and the function $\rho_i \in \Xi$ in case of $i = 1, \dots, n$.

Remark 4.1. Since we need this expression to converge to 0 as $n \to \infty$ this explains our choice of ε_n.

Averaging over all $y \in \Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}$ we obtain:

(2)
$$\left|\rho_i\left(H_{n-1} \circ g_n \circ D_{\psi_n,\gamma_n}^{-1}(x)\right) - \frac{1}{\mu\left(\Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}\right)} \int_{H_{n-1} \circ g_n\left(\Delta_{l,k,j_1^{(1)},j_2^{(1)},j_2^{(2)},j_2^{(3)}}\right)} \rho_i \, d\mu\right| < \frac{4}{n^8}$$

Furthermore we calculate that the trapping region $D_{\psi_n,\gamma_n}^{-1}\left(S_{l,k,j_1^{(1)},\vec{j_2}}^{\text{int}}\right)$ defined in section 2.2 is mapped under $\phi_n \circ D_{\psi_n,\gamma_n}$ onto

$$\bigcup \left[\frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2} + \frac{t_1^{(1)}}{n \cdot q_n^3} + ... + \frac{t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2}\right)} + 1}{n \cdot q_n^{2+3+4+...+(3+k-1)}} - \frac{t_2^{(1)}}{n \cdot q_n^{2+3+4+...+(3+k-1)+1}} - ... \right.$$
$$- \frac{t_2^{(3+k)} + 1}{n \cdot q_n^{2+3+4+...+(3+k-1)+3+k}} + \frac{t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)}}{n \cdot q_n^{2+3+...+(3+k-1)+(3+k)+1}} + ... + \frac{t_1^{\left(3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k\right)}}{\gamma_n} + \frac{1}{n^4 \cdot \gamma_n},$$
$$\left. \frac{l}{q_n} + \frac{k}{n \cdot q_n} + ... + \frac{t_1^{\left(3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k\right)} + 1}{\gamma_n} - \frac{1}{n^4 \cdot \gamma_n} \right]$$
$$\times \left[\frac{j_2^{(1)}}{q_n} + ... + \frac{j_2^{(3+k)}}{q_n^{3+k}} + \frac{\varepsilon_n}{q_n^{3+k}}, \frac{j_2^{(1)}}{q_n} + ... + \frac{j_2^{(3+k)} + 1}{q_n^{3+k}} - \frac{\varepsilon_n}{q_n^{3+k}} \right]$$

where the union is taken over all $t_2^{(l)} \in \mathbb{Z}$, $0 \leq t_2^{(l)} \leq q_n - 1$, for $l = 2, ..., 3+k$ as well as $t_2^{(1)} \in \mathbb{Z}$, $\lceil (4n^2+1)\, \varepsilon_n \cdot q_n \rceil \leq t_2^{(1)} \leq q_n - \lceil (4n^2+1)\, \varepsilon_n \cdot q_n \rceil - 1$ as well as $t_1^{(j)} \in \mathbb{Z}$, $0 \leq t_1^{(j)} \leq q_n - 1$, for $j = 1, ..., 3 \cdot (n-1) + \frac{n \cdot (n-1)}{2} - k$ apart from $t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)}$ satisfying $\lceil \varepsilon_n \cdot q_n \rceil \leq t_1^{\left(3 \cdot k + \frac{k \cdot (k-1)}{2} + 1\right)} \leq q_n - \lceil \varepsilon_n \cdot q_n \rceil - 1$.
In particular $\phi_n \circ D_{\psi_n, \gamma_n} \left(D^{-1}_{\psi_n, \gamma_n} \left(S^{\text{int}}_{l,k,j_1^{(1)}, \vec{j_2}} \right) \right)$ is contained in $\Delta_{l,k,j_1^{(1)} j_2^{(1)}, j_2^{(2)}, j_2^{(3)}}$. The same is true for the other allowed values of $j_2^{(4)}, ..., j_2^{(3+k)}$. Thus there are q_n^k trapping regions, that are mapped into $\Delta_{l,k,j_1^{(1)}, j_2^{(1)}, j_2^{(2)}, j_2^{(3)}}$ under $\phi_n \circ D_{\psi_n, \gamma_n}$. Hence we can estimate the number of $i \in \{0, ..., q_{n+1} - 1\}$ such that $\phi_n \circ D_{\psi_n, \gamma_n} \circ R^i_{\alpha_{n+1}}(x)$ is contained in $\Delta_{l,k,j_1^{(1)}, j_2^{(1)}, j_2^{(2)}, j_2^{(3)}}$ by $q_n^k \cdot \varpi^n_{\text{int}}(x) \cdot q_{n+1} \cdot \frac{1 - \frac{12}{n^2}}{n \cdot q_n^{3+k} \cdot q_n^2} = \varpi^n_{\text{int}}(x) \cdot q_{n+1} \cdot \left(1 - \frac{12}{n^2}\right) \cdot \mu \left(\Delta_{l,k,j_1^{(1)}, j_2^{(1)}, j_2^{(2)}, j_2^{(3)}} \right)$ from below and by $\varpi^n_{\text{int}}(x) \cdot q_{n+1} \cdot \mu \left(\Delta_{l,k,j_1^{(1)}, j_2^{(1)}, j_2^{(2)}, j_2^{(3)}} \right)$ from above for arbitrary $x \in \mathbb{S}^1 \times [0,1]$ using Remark 2.3.
Let $x \in \mathbb{S}^1 \times [0,1]$ be arbitrary. We denote the set of iterates $j \in \{0, ..., q_{n+1} - 1\}$ such that $\phi_n \circ D_{\psi_n, \gamma_n} \circ R^j_{\alpha_{n+1}}(x)$ is contained in $\Delta \in \tilde{T}^{\text{int}}_n$ by I_Δ. With the aid of equation 2 we obtain:

$$\left| \frac{1}{q_{n+1}} \sum_{j \in I_\Delta} \rho_i \left(H_{n-1} \circ g_n \circ D^{-1}_{\psi_n, \gamma_n} \circ \phi_n \circ D_{\psi_n, \gamma_n} \circ R^j_{\alpha_{n+1}}(x) \right) - \varpi^n_{\text{int}}(x) \cdot \int_{H_{n-1} \circ g_n(\Delta)} \rho_i d\mu \right|$$
$$\leq \frac{4 \cdot \mu(\Delta)}{n^8} + \frac{12}{n^2} \cdot \int_{H_{n-1} \circ g_n(\Delta)} |\rho_i| \, d\mu$$

Furthermore, we examine the trapping regions in the neighbourhoods of the boundaries. For $l = 0, 1, ..., q_n - 1$, $k = 0, 1, ..., n-1$ and $\lceil 18n^2 \varepsilon_n \cdot q_n \rceil \leq j_1^{(1)} \leq q_n - \lceil 18n^2 \varepsilon_n \cdot q_n \rceil - 1$ we introduce the sets

$$\Delta^0_{l,k,j_1^{(1)}} = \left[\frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2}, \frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)} + 1}{n \cdot q_n^2} \right] \times \left[0, 4 \cdot n^2 \cdot \varepsilon_n\right]$$

and

$$\Delta^1_{l,k,j_1^{(1)}} = \left[\frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2}, \frac{l}{q_n} + \frac{k}{n \cdot q_n} + \frac{j_1^{(1)} + 1}{n \cdot q_n^2} \right] \times \left[1 - 8 \cdot n^2 \cdot \varepsilon_n, 1\right].$$

Again $\tilde{T}^t_n$ denotes the collection of these sets $\Delta^t_{l,k,j_1^{(1)}}$ in case of $t=0,1$ as well as I_{Δ^0} and I_{Δ^1} respectively label the set of iterates such that $D_{\psi_n,\gamma_n}\circ R^j_{\alpha_{n+1}}(x)$ is contained in $\Delta^t\in\tilde{T}^t_n$ for $t=0$ and accordingly $t=1$.
We observe that for $t=0,1$ the map $H_{n-1}\circ g_n$ acts as the identity on these sets $\Delta^t_{l,k,j_1^{(1)}}$ and $\operatorname{diam}\left(\Delta^t_{l,k,j_1^{(1)}}\right)\leq 16\cdot n^2\cdot\varepsilon_n$. Then we conclude for $i=1,...,n$ and $x,y\in\Delta^t_{l,k,j_1^{(1)}}$

$$(3)\quad \left|\rho_i\left(H_{n-1}\circ g_n(x)\right)-\rho_i\left(H_{n-1}\circ g_n(y)\right)\right|<\operatorname{Lip}(\rho_i)\cdot\operatorname{diam}\left(\Delta^t_{l,k,j_1^{(1)}}\right)<\frac{4}{n^9\cdot q_{n-1}^{5\cdot r(n-1)}}<\frac{1}{n^8}$$

In particular this holds true for $y=\left(\frac{l}{q_n}+\frac{k}{n\cdot q_n}+\frac{j_1^{(1)}}{n\cdot q_n^2},t\right)$. We consider such points $\left(\frac{u}{n\cdot q_n^2},t\right)$ for $u=0,...,nq_n^2-1$ and calculate for $z\in\left[\frac{u}{n\cdot q_n^2}-\frac{1}{2\cdot n\cdot q_n^2},\frac{u}{n\cdot q_n^2}+\frac{1}{2\cdot n\cdot q_n^2}\right]$:

$$\left|\rho_i\left(\left(\frac{u}{n\cdot q_n^2},t\right)\right)-\rho_i\left((z,t)\right)\right|<\operatorname{Lip}(\rho_i)\cdot\frac{1}{2\cdot n\cdot q_n^2}<\frac{1}{n\cdot q_n}$$

Averaging over all $z\in\left[\frac{u}{n\cdot q_n^2}-\frac{1}{2\cdot n\cdot q_n^2},\frac{u}{n\cdot q_n^2}+\frac{1}{2\cdot n\cdot q_n^2}\right]$ yields

$$\left|\frac{1}{n\cdot q_n^2}\cdot\rho_i\left(\frac{u}{n\cdot q_n^2},t\right)-\int_{\left[\frac{u}{n\cdot q_n^2}-\frac{1}{2\cdot n\cdot q_n^2},\frac{u}{n\cdot q_n^2}+\frac{1}{2\cdot n\cdot q_n^2}\right]}\rho_i\,d\delta^t\right|<\frac{1}{n\cdot q_n}\cdot\frac{1}{n\cdot q_n^2}$$

Summing over all $u=0,...,nq_n^2-1$ gives

$$\left|\frac{1}{n\cdot q_n^2}\cdot\sum_{u=0}^{nq_n^2-1}\rho_i\left(\frac{u}{n\cdot q_n^2},t\right)-\int_{\mathbb{S}^1}\rho_i\,d\delta^t\right|<\frac{1}{n\cdot q_n}$$

The set of $u\in\left\{0,...,nq_n^2-1\right\}$ such that $\left(\frac{u}{n\cdot q_n^2},t\right)$ is contained in one of the blocks $\Delta^t_{l,k,j_1^{(1)}}\in\tilde{T}^t_n$ is denoted by U^t_n. Since there are at least $q_n\cdot n\cdot\left(q_n-2\cdot\lceil 18n^2\varepsilon_n\cdot q_n\rceil\right)\geq\left(1-\frac{1}{n^4}\right)\cdot n\cdot q_n^2$ such blocks there are at most $\lceil\frac{1}{n^4}\cdot n\cdot q_n^2\rceil$ numbers $u\in\left\{0,...,nq_n^2-1\right\}$ outside of U^t_n. Hereby we get

$$\begin{aligned}
&\left|\frac{1}{q_{n+1}}\sum_{\Delta^t\in\tilde{T}^t_n}\sum_{j\in I_{\Delta^t}}\rho_i\left(H_n\circ R^j_{\alpha_{n+1}}(x)\right)-\varpi^n_t(x)\cdot\int_{\mathbb{S}^1}\rho_i(\theta,t)\,d\delta^t\right|\\
\leq&\left|\frac{1}{q_{n+1}}\sum_{\Delta^t\in\tilde{T}^t_n}\sum_{j\in I_{\Delta^t}}\rho_i\left(H_n\circ R^j_{\alpha_{n+1}}(x)\right)-\varpi^n_t(x)\cdot\frac{1}{n\cdot q_n^2}\cdot\sum_{u=0}^{nq_n^2-1}\rho_i\left(\frac{u}{n\cdot q_n^2},t\right)\right|\\
&+\varpi^n_t(x)\cdot\left|\frac{1}{n\cdot q_n^2}\cdot\sum_{u=0}^{nq_n^2-1}\rho_i\left(\frac{u}{n\cdot q_n^2},t\right)-\int_{\mathbb{S}^1}\rho_i\,d\delta^t\right|\\
\leq&\sum_{u\in U^t_n}\left|\frac{1}{q_{n+1}}\sum_{j\in I_{\Delta^t_u}}\rho_i\left(H_n\circ R^j_{\alpha_{n+1}}(x)\right)-\frac{\varpi^n_t(x)}{n\cdot q_n^2}\cdot\rho_i\left(\frac{u}{n\cdot q_n^2},t\right)\right|+\frac{1}{n^2}\cdot\|\rho_i\|_0+\frac{1}{n\cdot q_n}
\end{aligned}$$

Under the map $H_n\left(q_n - 2\cdot\lceil 18n^2\varepsilon_n\cdot q_n\rceil\right)\cdot q_n^{2+k}$ trapping regions $D^{-1}_{\psi_n,\gamma_n}\left(S^t_{l,k,j_1^{(1)},\vec{j_2}}\right)$ are mapped into one such $\Delta^t_{l,k,j_1^{(1)}}$. Thus for arbitrary $x\in\mathbb{S}^1\times[0,1]$ the set $\Delta^t_{l,k,j_1^{(1)}}$ captures at least $\varpi^n_t(x)\cdot q_{n+1}\cdot\left(1-\frac{14}{n^2}\right)\cdot\frac{1}{n\cdot q_n^2}$ and at most $\varpi^n_t(x)\cdot q_{n+1}\cdot\frac{1}{n\cdot q_n^2}$ iterates $D^{-1}_{\psi_n,\gamma_n}\circ\phi_n\circ D_{\psi_n,\gamma_n}\circ R^j_{\alpha_{n+1}}(x)$ by Remark 2.3. Then we can estimate with the aid of equation 3

$$\left|\frac{1}{q_{n+1}}\sum_{j\in I_{\Delta^t_u}}\rho_i\left(H_n\circ R^j_{\alpha_{n+1}}(x)\right)-\frac{\varpi^n_t(x)}{n\cdot q_n^2}\rho_i\left(\frac{u}{n\cdot q_n^2},t\right)\right|<\frac{\varpi^n_t(x)}{n\cdot q_n^2}\cdot\frac{1}{n^8}+\frac{14}{n^2}\cdot\frac{\varpi^n_t(x)}{n\cdot q_n^2}\cdot\|\rho_i\|_0$$

In continuation of the above estimate we conclude

$$\begin{aligned}&\left|\frac{1}{q_{n+1}}\sum_{\Delta^t\in\tilde{T}^t_n}\sum_{j\in I_{\Delta^t}}\rho_i\left(H_n\circ R^j_{\alpha_{n+1}}(x)\right)-\varpi^n_t(x)\cdot\int_{\mathbb{S}^1}\rho_i(\theta,t)\,d\delta^t\right|\\ \leq&\frac{1}{n^8}+\frac{14}{n^2}\cdot\|\rho_i\|_0+\frac{1}{n^2}\cdot\|\rho_i\|_0+\frac{1}{n\cdot q_n}\leq\frac{15}{n^2}\cdot\|\rho_i\|_0+\frac{2}{n^8}\end{aligned}$$

Using this preparatory work we can prove the following result on the Birkhoff sums:

Lemma 4.2. *Let $\rho_i\in\Xi$ and $i=1,...,n$. Then for every $y\in M=\mathbb{S}^1\times[0,1]$ we have*

$$\inf_{\xi^n\in\Theta}\left|\frac{1}{q_{n+1}}\sum_{j=0}^{q_{n+1}-1}\rho_i\left(f_n^j(y)\right)-\int_M\rho_i\,d\xi^n\right|<\frac{60}{n^2}\cdot\|\rho_i\|_0$$

where Θ is the simplex generated by $\left\{\mu,\delta^0,\delta^1\right\}$.

Proof. Let $x\in\mathbb{S}^1\times[0,1]$ be arbitrary. We introduce the measure

$$\xi^n_x:=\varpi^n_{\text{int}}(x)\cdot\mu+\varpi^n_0(x)\cdot\delta^0+\varpi^n_1(x)\cdot\delta^1\in\Theta.$$

The set of numbers $k\in\{0,1,...,q_{n+1}-1\}$ such that the iterates $R^k_{\alpha_{n+1}}(x)$ are not contained in one of the trapping regions is denoted by I_a. Referred to Remark 2.4 there are at most $\frac{14}{n^2}\cdot q_{n+1}$ numbers in I_a. We obtain $\left|\sum_{j\in I_a}\rho_i\left(H_n\circ R^j_{\alpha_{n+1}}(x)\right)\right|\leq\|\rho_i\|_0\cdot\frac{14}{n^2}\cdot q_{n+1}$.

Hereby we obtain:

$$\left| \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \rho_i \left(H_n \circ R^j_{\alpha_{n+1}}(x) \right) - \varpi^n_{\text{int}}(x) \cdot \int_M \rho_i \, d\mu - \varpi^n_0(x) \cdot \int_{\mathbb{S}^1} \rho_i \, d\delta^0 - \varpi^n_1(x) \cdot \int_{\mathbb{S}^1} \rho_i \, d\delta^1 \right|$$

$$\leq \left| \frac{1}{q_{n+1}} \sum_{\Delta^{\text{int}} \in \tilde{T}^{\text{int}}_n} \sum_{j \in I_{\Delta^{\text{int}}}} \rho_i \left(H_n \circ R^j_{\alpha_{n+1}}(x) \right) - \varpi^n_{\text{int}}(x) \cdot \int_M \rho_i \, d\mu \right|$$

$$+ \left| \frac{1}{q_{n+1}} \sum_{\Delta^0 \in \tilde{T}^0_n} \sum_{j \in I_{\Delta^0}} \rho_i \left(H_n \circ R^j_{\alpha_{n+1}}(x) \right) - \varpi^n_0(x) \cdot \int_{\mathbb{S}^1} \rho_i(\theta, 0) \, d\delta^0 \right|$$

$$+ \left| \frac{1}{q_{n+1}} \sum_{\Delta^1 \in \tilde{T}^1_n} \sum_{j \in I_{\Delta^1}} \rho_i \left(H_n \circ R^j_{\alpha_{n+1}}(x) \right) - \varpi^n_1(x) \cdot \int_{\mathbb{S}^1} \rho_i(\theta, 1) \, d\delta^1 \right|$$

$$+ \left| \frac{1}{q_{n+1}} \sum_{j \in I_a} \rho_i \left(H_n \circ R^j_{\alpha_{n+1}}(x) \right) \right|$$

$$\leq \frac{4}{n^8} + \frac{12}{n^2} \cdot \|\rho_i\|_0 + \mu\left(M \setminus T^{\text{int}}_n\right) \cdot \|\rho_i\|_0 + 2 \cdot \left(\frac{15}{n^2} \cdot \|\rho_i\|_0 + \frac{2}{n^8} \right) + \frac{14}{n^2} \cdot \|\rho_i\|_0 \leq \frac{60}{n^2} \cdot \|\rho_i\|_0$$

With $x = H_n^{-1}(y)$ we obtain the statement of the Lemma. □

We point out that the measure ξ^n_x used in the above proof was dependent on the point x, but independent of the function $\rho \in \Xi$.

Lemma 4.3. *For every $\rho \in \Xi$ and $y \in \mathbb{S}^1 \times [0,1]$ we have*

$$\inf_{\xi^n \in \Theta} \left| \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho\left(f^k(y)\right) - \int \rho \, d\xi^n \right| \to 0 \quad \text{as } n \to \infty$$

where Θ is the simplex generated by $\left\{\mu, \delta^0, \delta^1\right\}$.

Proof. By Lemma 3.11 we have

$$d_0^{(q_{n+1})}(f, f_n) := \max_{i=0,1,\dots,q_{n+1}-1} d_0\left(f^i, f^i_n\right) \overset{n\to\infty}{\to} 0$$

Then for every $\rho \in \Xi$ we have $\left|\rho\left(f^i(x)\right) - \rho\left(f^i_n(x)\right)\right| \overset{n\to\infty}{\to} 0$ uniformly for $i = 0, 1, \dots, q_{n+1}-1$, because every continuous function on the compact space $\mathbb{S}^1 \times [0,1]$ is uniformly continuous . Thus we get: $\left\| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \rho\left(f^i(x)\right) - \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \rho\left(f^i_n(x)\right) \right\|_0 \overset{n\to\infty}{\to} 0$. Applying the previous Lemma 4.2 we obtain the claim. □

Since the family Ξ is dense in $C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$ the convergence holds for every continuous function by an approximation argument.

Now we can prove that the measures μ, δ^0, δ^1 are the only possible ergodic ones: Assume that there is another ergodic invariant probability measure ξ. By the Birkhoff Ergodic Theorem we have for every $\rho \in C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho\left(f^k(x)\right) = \int_{\mathbb{S}^1 \times [0,1]} \rho \, d\xi \quad \text{for } \xi\text{-a.e. } x \in \mathbb{S}^1 \times [0,1]$$

With the aid of Lemma 4.3 we obtain for every $\rho \in C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$ and x in a set of ξ-full measure:

$$\int_{\mathbb{S}^1 \times [0,1]} \rho \, d\xi = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \rho\left(f^k(x)\right) = \lim_{n\to\infty} \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho\left(f^k(x)\right) = \lim_{n\to\infty} \int_{\mathbb{S}^1 \times [0,1]} \rho \, d\xi^n,$$

where ξ^n is in the simplex generated by $\left\{\mu, \delta^0, \delta^1\right\}$. As noted this measure does not depend on the function ρ. Thus we have for every $\rho \in C\left(\mathbb{S}^1 \times [0,1], \mathbb{R}\right)$: $\lim_{n\to\infty} \int_{\mathbb{S}^1 \times [0,1]} \rho \, d\xi^n = \int_{\mathbb{S}^1 \times [0,1]} \rho \, d\xi$. Since the simplex generated by $\left\{\mu, \delta^0, \delta^1\right\}$ is weakly closed this implies that ξ is in this simplex. We recall that ergodic measures are the extreme points in the set of invariant Borel probability measures (see [Wa75], Theorem 5.15.). Then ξ has to be one of the measures $\left\{\mu, \delta^0, \delta^1\right\}$ and we obtain a contradiction.

4.2 Weak mixing with respect to Lebesgue measure on $\mathbb{S}^1 \times [0,1]$

We introduce the central notion in the proof of the weak mixing-property:

Definition 4.4. Let $\Phi : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$ be a diffeomorphism and J be an interval in $[0,1]$. We say that an element $\hat{I}$ of a partial partition is (γ, ϵ)-distributed on J under Φ, if the following properties are satisfied:

- $[c, c+\tilde{\gamma}] \times J \subseteq \Phi\left(\hat{I}\right) \subseteq [c, c+\tilde{\gamma}] \times [0,1]$ for some $c \in \mathbb{S}^1$ and $\tilde{\gamma} \leq \gamma$
- For every interval $\tilde{J} \subseteq J$ it holds:

$$\left| \frac{\mu\left(\hat{I} \cap \Phi^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}\right)} - \frac{\lambda\left(\tilde{J}\right)}{\lambda(J)} \right| \leq \epsilon \cdot \frac{\lambda\left(\tilde{J}\right)}{\lambda(J)}$$

Remark 4.5. Analogous to [FS05] we will call the second property "almost uniform distribution" of $\hat{I}$ on J. In the following we will often write it in the form

$$\left|\mu\left(\hat{I} \cap \Phi^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right) \cdot \lambda(J) - \mu\left(\hat{I}\right) \cdot \lambda\left(\tilde{J}\right)\right| \leq \epsilon \cdot \mu\left(\hat{I}\right) \cdot \lambda\left(\tilde{J}\right)$$

In the next step we define the sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$:

$$\begin{aligned} m_n &= \min\left\{ m \leq q_{n+1} \;:\; m \in \mathbb{N},\; \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} + \frac{k}{q_n} \right| \leq \frac{1}{\varepsilon_{n+1} \cdot q_{n+1}} \right\} \\ &= \min\left\{ m \leq q_{n+1} \;:\; m \in \mathbb{N},\; \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{n} + k \right| \leq \frac{q_n}{\varepsilon_{n+1} \cdot q_{n+1}} \right\} \end{aligned}$$

Lemma 4.6. *The set* $M_n := \left\{ m \leq q_{n+1} \;:\; m \in \mathbb{N},\; \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{n} + k \right| \leq \frac{q_n}{\varepsilon_{n+1} \cdot q_{n+1}} \right\}$ *is non-empty for every* $n \in \mathbb{N}$, *i.e.* m_n *exists.*

Proof. In Lemma 3.8 we constructed the sequence $\alpha_n = \frac{p_n}{q_n}$ in such a way that $q_n = \frac{1}{\varepsilon_n} \cdot \tilde{q}_n$ and $p_n = \frac{1}{\varepsilon_n} \cdot \tilde{p}_n$ with $\tilde{p}_n, \tilde{q}_n$ relatively prime. Therefore the set $\left\{ j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} \;:\; j = 1, 2, ..., q_{n+1} \right\}$ contains $\frac{\varepsilon_{n+1} \cdot q_{n+1}}{\gcd(q_n, \tilde{q}_{n+1})}$ different equally distributed points on $\mathbb{S}^1$. Hence there are at least $\frac{\varepsilon_{n+1} \cdot q_{n+1}}{q_n}$ different such points and so for every $x \in \mathbb{S}^1$ there is a $j \in \{1, ..., q_{n+1}\}$, such that

$$\inf_{k\in\mathbb{Z}} \left| x - j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} + k \right| \leq \frac{q_n}{\varepsilon_{n+1} \cdot q_{n+1}}.$$

In particular this is true for $x = \frac{1}{n}$. $\square$

Remark 4.7. We define

$$a_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n}\right) \mod \frac{1}{q_n}$$

By the above construction of m_n it holds: $|a_n| \leq \frac{1}{\varepsilon_{n+1} \cdot q_{n+1}}$. In the proof of Lemma 3.8 we set the condition $q_{n+1} \geq \frac{1}{\varepsilon_{n+1}} \cdot 12 \cdot \frac{1}{\varepsilon_n} \cdot |||\psi_n|||_1 \cdot \gamma_n^2$. Thus we get:

$$|a_n| \leq \frac{\varepsilon_n}{12 \cdot |||\psi_n|||_1 \cdot \gamma_n^2}$$

Our constructions are done in such a way that the following property is satisfied:

Lemma 4.8. *We consider the interval* $J := \left[25n^2 \cdot \varepsilon_n, 1 - 25n^2 \cdot \varepsilon_n\right]$ *as well as the diffeomorphism* $\Phi_n := D^{-1}_{\psi_n,\gamma_n} \circ \phi_n \circ D_{\psi_n,\gamma_n} \circ R^{m_n}_{\alpha_{n+1}} \circ D^{-1}_{\psi_n,\gamma_n} \circ \phi_n^{-1}$ *with the conjugating map* ϕ_n *defined in section 2.5. Then the elements of the partition* η_n *are* $\left(\frac{1}{n \cdot q_n^2}, \frac{1}{n}\right)$*-distributed on* J *under* Φ_n*.*

In order to prove the weak mixing property we modify the proof from [Kun13b], section 5. We recall the following approximation statement ([Kun13b], Lemma 5.2):

Lemma 4.9. *Let* $f = \lim_{n\to\infty} f_n$ *be a diffeomorphism obtained by the constructions in the preceding sections and* $(m_n)_{n\in\mathbb{N}}$ *be a sequence of natural numbers fulfilling* $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$*. Furthermore let* $(\nu_n)_{n\in\mathbb{N}}$ *be a sequence of partial partitions, where* $\nu_n \to \varepsilon$ *and for every* $n \in \mathbb{N}$ ν_n *is the image of a partial partition* η_n *under a measure-preserving diffeomorphism* F_n*, satisfying the following property: For every cube* $A \subseteq \mathbb{S}^1 \times (0,1)$ *and for every* $\epsilon \in (0,1]$ *there exists* $N \in \mathbb{N}$ *such that for every* $n \geq N$ *we have for every* $\Gamma_n \in \nu_n$

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A)\right| \leq \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A) \tag{4}$$

Then f *is weak mixing.*

In our case we will use the subsequent sequence of partial partitions and we will need that it converges to the decomposition into points.

Lemma 4.10. *Consider the sequence of partial partitions* $(\eta_n)_{n\in\mathbb{N}}$ *constructed in section 2.3.1. Furthermore, let* $(H_n)_{n\in\mathbb{N}}$ *be a sequence of measure-preserving smooth diffeomorphisms satisfying* $\|DH_{n-1}\| \leq \frac{\ln(q_n)}{n}$ *for every* $n \in \mathbb{N}$ *and define the partial partitions*

$$\nu_n = \left\{\Gamma_n = H_{n-1} \circ g_n \circ D^{-1}_{\psi_n,\gamma_n}\left(\hat{I}_n\right) \ : \ \hat{I}_n \in \eta_n\right\}$$

Then we get $\nu_n \to \varepsilon$*.*

Proof. Since the trapping map $D^{-1}_{\psi_n,\gamma_n}$ causes a r-translation by at most $4n^2 \cdot \varepsilon_n$ we have $D^{-1}_{\psi_n,\gamma_n}\left(\hat{I}_n\right) \subseteq \mathbb{S}^1 \times \left[12n^2 \cdot \varepsilon_n, 1 - 12n^2 \cdot \varepsilon_n\right]$ due to the choice of $j_2^{(1)}$.
After the application of $D^{-1}_{\psi_n,\gamma_n}$ on $\hat{I}_n \in \eta_n$ the diameter is at most $\sqrt{2} \cdot \left(\frac{1}{q_n^5} + 4n^2 \cdot \varepsilon_n\right) \leq 2 \cdot \sqrt{2} \cdot 4n^2 \varepsilon_n$. Unfortunately, on this set $g_n = \tilde{g}_{[nq_n^{\sigma_n}]}$ is not necessarily true, but it is strictly contained in such a cube of sidelength $2 \cdot \sqrt{2} \cdot 4n^2 \varepsilon_n$ that is a union of domains of $g_{a,b,\varepsilon}$. Under the above assumption $q_n > n^{13} \cdot q_{n-1}^{9 \cdot r(n-1)+1}$ we obtain for the diameter of such a partition element:

$$\begin{aligned} \operatorname{diam}\left(H_{n-1} \circ g_n \circ D^{-1}_{\psi_n,\gamma_n}\left(\hat{I}_n\right)\right) &\leq \|DH_{n-1}\|_0 \cdot [nq_n^{\sigma_n}] \cdot 2 \cdot \sqrt{2} \cdot 4n^2 \cdot \varepsilon_n \\ &\leq q_{n-1}^{r(n-1)} \cdot n \cdot q_{n-1}^{4 \cdot r(n-1)} \cdot \frac{2 \cdot \sqrt{2}}{n^9 \cdot q_{n-1}^{5 \cdot r(n-1)+1}} \to 0 \end{aligned}$$

as $n \to \infty$. Thus this sequence of partial partitions converges to the decomposition into points. $\square$

As a technical result needed in the proof of Lemma 4.12 we state [Kun13b], Lemma 5.4.:

Lemma 4.11. *Given an interval on the r-axis of the form $K = \bigcup_{k\in\mathbb{Z}, k_1\leq k\leq k_2}\left[\frac{k\cdot\varepsilon}{b\cdot a}, \frac{(k+1)\cdot\varepsilon}{b\cdot a}\right]$, where $k_1, k_2 \in \mathbb{Z}$ with $\frac{b\cdot a}{\varepsilon}\cdot\delta \leq k_1 < k_2 \leq \frac{b\cdot a}{\varepsilon} - \frac{b\cdot a}{\varepsilon}\cdot\delta - 1$. $K_{c,\gamma}$ denotes the cuboid $[c, c+\gamma]\times K$ for some $\gamma > 0$. We consider the diffeomorphism $g_{a,b,\varepsilon}$ constructed in subsection 2.4 and an interval $L = [l_1, l_2]$ of $\mathbb{S}^1$ satisfying $\tilde{\lambda}(L) \geq 4\cdot\frac{1-2\varepsilon}{a} - \gamma$.*
If $b\cdot\lambda(K) > 2$. then for the set $Q := \pi_{\vec{r}}\left(K_{c,\gamma}\cap g_{a,b,\varepsilon}^{-1}(L\times K\times Z)\right)$ we have:

$$\left|\lambda(Q) - \lambda(K)\cdot\tilde{\lambda}(L)\right| \leq \frac{2}{b}\cdot\tilde{\lambda}(L) + \frac{2\cdot\gamma}{b} + \gamma\cdot\lambda(K) + 4\cdot\frac{1-2\varepsilon}{a}\cdot\lambda(K) + 8\cdot\frac{1-2\varepsilon}{b\cdot a}$$

Lemma 4.12. *Let $n \geq 5$. For the number m_n as above we consider*

$$\Phi_n = D^{-1}_{\psi_n,\gamma_n}\circ\phi_n\circ D_{\psi_n,\gamma_n}\circ R^{m_n}_{\alpha_{n+1}}\circ D^{-1}_{\psi_n,\gamma_n}\circ\phi_n^{-1}$$

and $J := \left[25n^2\cdot\varepsilon_n, 1-25n^2\cdot\varepsilon_n\right]$.
Then for every cube S of side length $q_n^{-\sigma_n}$ lying in $\mathbb{S}^1\times J$ we get

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\circ g_n^{-1}(S)\right)\cdot\lambda(J) - \mu\left(\hat{I}\right)\cdot\mu(S)\right| \leq \frac{21}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S) \tag{5}$$

Proof. Let S be a cube with sidelength $q_n^{-\sigma_n}$ lying in $\mathbb{S}^1\times J$. Furthermore we denote $S_\theta = \pi_\theta(S)$ and $S_r = \pi_r(S)$. Obviously: $\tilde{\lambda}(S_\theta) = \lambda(S_r) = q_n^{-\sigma_n}$ and $\tilde{\lambda}(S_\theta)\cdot\lambda(S_r) = \mu(S) = q_n^{-2\sigma_n}$.
According to Lemma 4.8 Φ_n $\left(\frac{1}{n\cdot q_n^2}, \frac{1}{n}\right)$-distributes the partition element $\hat{I}_n \in \eta_n$ on J, in particular $\Phi_n\left(\hat{I}_n\right) \subseteq [c, c+\gamma]\times[0,1]$ for some $c\in\mathbb{S}^1$ and some $\gamma \leq \frac{1}{n\cdot q_n^2}$. Furthermore we saw in the proof of Lemma 4.8 that $\phi_n\circ D_{\psi_n,\gamma_n}\circ R^{m_n}_{\alpha_{n+1}}\circ D^{-1}_{\psi_n,\gamma_n}\circ\phi_n^{-1}\left(\hat{I}_n\right)$ is contained in the interior of the step-by-step domains of the map g_n and on its boundary $g_n = \tilde{g}_{[nq_n^{\sigma_n}]}$ holds. Particularly it follows $\gamma \geq \frac{1-2\varepsilon}{a}$ in case of $g_n = g_{a,b,\varepsilon}$. For $l\in\mathbb{Z}$, $0\leq l\leq\frac{b\cdot a}{\varepsilon} - 1$ we introduce the sets $\Delta_l = \left[\frac{l\varepsilon}{ba}, \frac{(l+1)\varepsilon}{ba}\right]$ and with these we consider

$$\tilde{S}_r := \bigcup_{\Delta_l\subseteq S_r}\Delta_l \qquad \text{as well as} \qquad \tilde{S} := S_\theta\times\tilde{S}_r\subseteq S$$

Using the triangle inequality we obtain

$$\begin{aligned}\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right)\cdot\lambda(J) - \mu\left(\hat{I}\right)\cdot\mu(S)\right| &\leq \left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right|\cdot\lambda(J)\\ &+\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\cdot\lambda(J) - \mu\left(\hat{I}\right)\mu\left(\tilde{S}\right)\right| + \mu\left(\hat{I}\right)\cdot\left|\mu\left(\tilde{S}\right) - \mu(S)\right|\end{aligned}$$

Here $\left|\mu\left(\tilde{S}\right) - \mu(S)\right| = \mu\left(S\setminus\tilde{S}\right) \leq 2\cdot\frac{\varepsilon}{b\cdot a}\cdot\tilde{\lambda}(S_\theta) \leq 2\cdot\frac{\varepsilon}{a}\cdot\mu(S)$, where we used $b = [n\cdot q_n^{\sigma_n}] \geq q_n^{\sigma_n}$ in case of $n > 4$. Since Φ_n and g_n are measure-preserving we obtain additionally:

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| \leq \mu\left(S\setminus\tilde{S}\right) \leq 2\cdot\frac{\varepsilon}{a}\cdot\mu(S)$$

In the proof of Lemma 4.8 we observed $\mu\left(\Phi_n\left(\hat{I}\right)\right) \geq \frac{1}{a}\cdot\left(1-\frac{2}{26n^4}\right)\cdot\lambda\left(J\right)$. Hence:

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right)-\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right|\cdot\lambda\left(J\right)\leq 2\cdot\frac{\varepsilon}{a}\cdot\mu\left(S\right)\cdot\lambda\left(J\right)$$
$$\leq 2\cdot\frac{\varepsilon}{1-\frac{2}{26n^4}}\cdot\mu\left(S\right)\cdot\mu\left(\Phi_n\left(\hat{I}\right)\right)\leq 4\cdot\varepsilon\cdot\mu\left(S\right)\cdot\mu\left(\Phi_n\left(\hat{I}\right)\right)=4\cdot\varepsilon\cdot\mu\left(S\right)\cdot\mu\left(\hat{I}\right)$$

Thus we obtain:

$$\begin{aligned}&\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right)\cdot\lambda\left(J\right)-\mu\left(\hat{I}\right)\cdot\mu\left(S\right)\right|\\ &\leq\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\cdot\lambda\left(J\right)-\mu\left(\hat{I}\right)\mu\left(\tilde{S}\right)\right|+5\cdot\varepsilon\cdot\mu\left(S\right)\cdot\mu\left(\hat{I}\right)\end{aligned}\tag{6}$$

Next, we want to estimate the first summand. By construction of the map $g_n = g_{a,b,\varepsilon}$ and the definition of $\tilde{S}$ it holds: $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq[c,c+\gamma]\times\tilde{S}_r =: K_{c,\gamma}$. Considering the proof of Lemma 4.8 again, we obtain $g_n\left(K_{c,\gamma}\right)=\tilde{g}_{[nq_n^{\sigma_n}]}\left(K_{c,\gamma}\right)$ (since c and $c+\gamma$ are in the domain where $g_n=\tilde{g}_{[nq_n^{\sigma_n}]}$ holds).
Because of Lemma 4.8: $2\gamma\leq\frac{2}{n\cdot q_n^2}<q_n^{-\sigma_n}$ for $n>2$. So we can define a cuboid $S_1\subseteq\tilde{S}$, where $S_1:=[s_1+\gamma,s_2-\gamma]\times\tilde{S}_r$ using the notation $S_\theta=[s_1,s_2]$. We examine the two sets

$$Q:=\pi_r\left(K_{c,\gamma}\cap g_n^{-1}\left(S_\theta\times\tilde{S}_r\right)\right)\qquad\qquad Q_1:=\pi_r\left(K_{c,\gamma}\cap g_n^{-1}\left([s_1+\gamma,s_2-\gamma]\times\tilde{S}_r\right)\right)$$

As seen above $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq K_{c,\gamma}$. Hence $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\cap K_{c,\gamma}$, which implies $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q\right)$.

Claim: On the other hand: $\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q_1\right)\subseteq\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)$.

Proof of the claim: For $(\theta,r)\in\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q_1\right)$ arbitrary it holds $(\theta,r)\in\Phi_n\left(\hat{I}\right)$, i.e. $\theta\in[c,c+\gamma]$, and $r\in\pi_r\left(K_{c,\gamma}\cap g_n^{-1}\left([s_1+\gamma,s_2-\gamma]\times\tilde{S}_r\right)\right)$, i.e. in particular $r\in\tilde{S}_r$. This implies the existence of $\bar{\theta}\in[c,c+\gamma]$ satisfying $\left(\bar{\theta},r\right)\in K_{c,\gamma}\cap g_n^{-1}\left(S_1\right)$. Hence there are $\beta\in[s_1+\gamma,s_2-\gamma]$ and $r_1\in\tilde{S}_r$, such that $g_n\left(\bar{\theta},r\right)=\left(\beta,r_1\right)$. Because of $\bar{\theta}\in[c,c+\gamma]$ and $r\in\tilde{S}_r$ the point $\left(\bar{\theta},r\right)$ is contained in one cuboid of the form $\Delta_{a,b,\varepsilon}$. Since $\theta\in[c,c+\gamma]$ (θ,r) is contained in the same $\Delta_{a,b,\varepsilon}$. Thus $\pi_r\left(g_n\left(\theta,r\right)\right)\in\tilde{S}_r$. Furthermore $g_n\left(\theta,r\right)$ and $g_n\left(\bar{\theta},r\right)$ are in a distance of at most γ on the θ-axis, because $\theta,\bar{\theta}\in[c,c+\gamma]$, $g_n\left(K_{c,\gamma}\right)=\tilde{g}_{[nq_n^{\sigma_n}]}\left(K_{c,\gamma}\right)$ and the map $\tilde{g}_{[nq_n^{\sigma_n}]}$ preserves the distances on the θ-axis. Thus there are $\bar{\beta}\in[s_1,s_2]$ and $r_2\in\tilde{S}_r$ such that $g_n\left(\theta,r\right)=\left(\bar{\beta},r_2\right)$. So $(\theta,r)\in\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)$. □

Altogether the following inclusions are true:

$$\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q_1\right)\subseteq\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q\right)$$

Thus we obtain:

$$\begin{aligned}&\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(\tilde{S})\right)\right)\cdot\lambda\left(J\right)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|\\ &\leq\max\Bigg(\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)\cdot\lambda\left(J\right)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|,\\ &\qquad\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q_1\right)\right)\cdot\lambda\left(J\right)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|\Bigg)\end{aligned}\tag{7}$$

We want to apply Lemma 4.11 for $K=\tilde{S}_r$, $L=S_\theta$ and $b=[n\cdot q_n^{\sigma_n}]$ (note that the requirements $4\cdot\frac{1-2\varepsilon}{a}-\gamma\leq 3\cdot\frac{1-2\varepsilon}{a}<\frac{1}{q_n^{\sigma_n}}=\tilde{\lambda}(L)$ and $b\cdot\lambda(K)=[nq_n^{\sigma_n}]\cdot q_n^{-\sigma_n}\geq\frac{1}{2}nq_n^{\sigma_n}\cdot q_n^{-\sigma_n}>2$ for $n>4$ are fulfilled):

$$\begin{aligned}
&\left|\lambda(Q)-\mu\left(\tilde{S}\right)\right| \\
&\leq \frac{2}{[n\cdot q_n^{\sigma_n}]}\cdot\tilde{\lambda}(S_\theta)+\frac{2\gamma}{[n\cdot q_n^{\sigma_n}]}+\gamma\cdot\lambda\left(\tilde{S}_r\right)+4\cdot\frac{1-2\varepsilon}{a}\lambda\left(\tilde{S}_r\right)+8\cdot\frac{1-2\varepsilon}{[nq_n^{\sigma_n}]\cdot a} \\
&\leq \frac{4}{n\cdot q_n^{\sigma_n}}\cdot\tilde{\lambda}(S_\theta)+\frac{4}{n\cdot q_n^{\sigma_n}\cdot q_n^{\sigma_n}}+\frac{1}{n\cdot q_n^{\sigma_n}}\cdot\lambda(S_r)+4\cdot\frac{1-2\varepsilon}{n\cdot q_n^2}\cdot\lambda(S_r)+\frac{16\cdot(1-2\varepsilon)}{n\cdot q_n^{\sigma_n}\cdot n\cdot q_n^2} \\
&\leq \frac{14}{n}\cdot\mu(S)
\end{aligned}$$

In particular we receive from this estimate: $\frac{14}{n}\cdot\mu(S)\geq\lambda(Q)-\mu\left(\tilde{S}\right)\geq\lambda(Q)-\mu(S)$, hence: $\lambda(Q)\leq\left(1+\frac{14}{n}\right)\cdot\mu(S)\leq 4\cdot\mu(S)$.
Analogously we obtain: $\lambda(Q_1)\leq 4\cdot\mu(S)$.
Since Q as well as Q_1 are a finite union of disjoint intervals contained in J and Φ_n $\left(\frac{1}{n\cdot q_n^2},\frac{1}{n}\right)$-distributes the interval $\hat{I}$ on J we get:

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\lambda(Q)\right|\leq\frac{1}{n}\cdot\mu\left(\hat{I}\right)\cdot\lambda(Q)\leq\frac{4}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)$$

as well as

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q_1\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\lambda(Q_1)\right|\leq\frac{1}{n}\cdot\mu\left(\hat{I}\right)\cdot\lambda(Q_1)\leq\frac{4}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)$$

Now we can proceed

$$\begin{aligned}
&\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right| \\
&\leq\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\lambda(Q)\right|+\mu\left(\hat{I}\right)\cdot\left|\lambda(Q)-\mu\left(\tilde{S}\right)\right| \\
&\leq\frac{4}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)+\mu\left(\hat{I}\right)\cdot\frac{14}{n}\cdot\mu(S)=\frac{18}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)
\end{aligned}$$

Noting that $\mu(S_1)=\mu\left(\tilde{S}\right)-2\gamma\cdot\lambda\left(\tilde{S}_r\right)$ and so $\mu\left(\tilde{S}\right)-\mu(S_1)\leq 2\cdot\frac{1}{n\cdot q_n^{\sigma_n}}\cdot\lambda\left(\tilde{S}_r\right)\leq\frac{2}{n}\cdot\mu(S)$ we obtain in the same way as above:

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q_1\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|\leq\frac{20}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)$$

Using equation 7 this yields:

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|\leq\frac{20}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)$$

Finally we conclude with the aid of equation 6 because of $\varepsilon=\frac{1}{8n^4}$:

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right)\cdot\lambda(J)-\mu\left(\hat{I}\right)\cdot\mu(S)\right|\leq\frac{21}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu(S)$$

□

Now we are able to prove the aimed weak mixing property:

Proposition 4.13. *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n\in\mathbb{N}}$ be constructed as above. Suppose additionally that $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ for every $n \in \mathbb{N}$, $\|DH_{n-1}\|_0 \leq \frac{\ln(q_n)}{n}$ and that the limit $f = \lim_{n\to\infty} f_n$ exists.*
Then f is weak mixing.

Proof. To apply Lemma 4.9 we consider the partial partitions $\nu_n := H_{n-1} \circ g_n \circ D^{-1}_{\psi_n,\gamma_n}(\eta_n)$. As proven in Lemma 4.10 these partial partitions satisfy $\nu_n \to \varepsilon$. We have to establish equation 4. For it let $\epsilon > 0$ and a cube $A \subseteq \mathbb{S}^1 \times (0,1)$ be given. There exists $N \in \mathbb{N}$ such that $A \subseteq \mathbb{S}^1 \times \left[25n^2 \cdot \varepsilon_n, 1 - 25n^2 \cdot \varepsilon_n\right]$ for every $n \geq N$. Because of Lemma 4.8 we obtain for every $\hat{I}_n \in \eta_n$: $\Phi_n\left(\hat{I}_n\right) \supseteq [c, c+\gamma] \times \left[25n^2 \cdot \varepsilon_n, 1 - 25n^2 \cdot \varepsilon_n\right]$ for some $\gamma \leq \frac{1}{n \cdot q_n^2}$. Furthermore, we note $f_n^{m_n} = H_n \circ R^{m_n}_{\alpha_{n+1}} \circ H_n^{-1} = H_{n-1} \circ g_n \circ \Phi_n \circ D_{\psi_n,\gamma_n} \circ g_n^{-1} \circ H_{n-1}^{-1}$.
Let S_n be a cube of sidelength $q_n^{-\sigma_n}$ contained in $\mathbb{S}^1 \times \left[25n^2 \cdot \varepsilon_n, 1 - 25n^2 \cdot \varepsilon_n\right] = \mathbb{S}^1 \times J$. We look at $C_n := H_{n-1}(S_n)$, $\Gamma_n \in \nu_n$, and compute (since g_n and H_{n-1} are measure-preserving):

$$\begin{aligned}
&\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n) \cdot \mu(C_n)\right| = \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)\right) - \mu\left(\hat{I}_n\right) \cdot \mu(S_n)\right| \\
&\leq \frac{1}{\lambda(J)} \cdot \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)\right) \cdot \lambda(J) - \mu\left(\hat{I}_n\right) \cdot \mu(S_n)\right| + \frac{1-\lambda(J)}{\lambda(J)} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n)
\end{aligned}$$

Since $\lambda(J) \geq \frac{1}{2}$ and so: $\frac{1-\lambda(J)}{\lambda(J)} \leq 2 \cdot (1 - \lambda(J)) \leq \frac{2}{n}$. We continue by applying Lemma 4.12:

$$\begin{aligned}
\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n) \cdot \mu(C_n)\right| &\leq 2 \cdot \frac{21}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n) + \frac{2}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n) \\
&= \frac{44}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n)
\end{aligned}$$

Moreover, it holds $\operatorname{diam}(C_n) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}(S_n) \leq q_{n-1}^{r(n-1)} \cdot \frac{\sqrt{2}}{q_n^{\sigma_n}} = q_{n-1}^{r(n-1)} \cdot \frac{\sqrt{2}}{q_{n-1}^{4\cdot r(n-1)}}$, i.e. $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$. Thus we can approximate A by a countable disjoint union of sets $C_n = H_{n-1}(S_n)$ with $S_n \subseteq \mathbb{S}^1 \times \left[25n^2 \cdot \varepsilon_n, 1 - 25n^2 \cdot \varepsilon_n\right]$ a cube of sidelength $q_n^{-\sigma_n}$ in given precision, when n is chosen big enough. Consequently for n sufficiently large there are sets $A_1 = \bigcup_{i\in\Sigma_n^1} C_n^i$ and $A_2 = \bigcup_{i\in\Sigma_n^2} C_n^i$ with countable sets Σ_n^1 and Σ_n^2 of indices satisfying $A_1 \subseteq A \subseteq A_2$ as well as $|\mu(A) - \mu(A_i)| \leq \frac{\epsilon}{3} \cdot \mu(A)$ for $i = 1, 2$.
Additionally we choose n such that $\frac{44}{n} < \frac{\epsilon}{3}$ holds. It follows:

$$\begin{aligned}
&\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A) \\
\leq&\mu\left(\Gamma_n \cap f_n^{-m_n}(A_2)\right) - \mu(\Gamma_n) \cdot \mu(A_2) + \mu(\Gamma_n) \cdot (\mu(A_2) - \mu(A)) \\
\leq& \sum_{i\in\Sigma_n^2} \left(\mu\left(\Gamma_n \cap f_n^{-m_n}\left(C_n^i\right)\right) - \mu(\Gamma_n) \cdot \mu\left(C_n^i\right)\right) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) \\
\leq& \sum_{i\in\Sigma_n^2} \left(\frac{44}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu\left(S_n^i\right)\right) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) \\
=&\frac{44}{n} \cdot \mu(\Gamma_n) \cdot \mu\left(\bigcup_{i\in\Sigma_n^2} C_n^i\right) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) \leq \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A_2) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) \\
=&\frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot (\mu(A_2) - \mu(A)) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) \leq \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A)
\end{aligned}$$

Analogously we estimate: $\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A) \geq -\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A)$. Both estimates enable us to conclude: $\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\right| \leq \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A)$. □

By Lemma 3.11 the requirement of the proximity between f and f_n is fulfilled. Hence f is weak mixing.

5 Construction of the f-invariant measurable Riemannian metric

Let ω_0 denote the standard Riemannian metric on $M = \mathbb{S}^1 \times [0,1]$. The following Lemma shows that the conjugation map $h_n = g_n \circ D^{-1}_{\psi_n,\gamma_n} \circ \phi_n \circ D_{\psi_n,\gamma_n}$ constructed in section 2 is an isometry with respect to ω_0 on the elements of the partial partition ζ_n.

Lemma 5.1. *Let $D^{-1}_{\psi_n,\gamma_n}\left(\hat{I}_n\right) \in \zeta_n$. Then $h_n|_{D^{-1}_{\psi_n,\gamma_n}(\hat{I}_n)}$ is an isometry with respect to ω_0.*

Proof. As noted in Remark 2.7 D_{ψ_n,γ_n} acts as an isometry on any element $D^{-1}_{\psi_n,\gamma_n}\left(\check{I}_n\right) \in \zeta_n$. Next we observe that ϕ_n is an isometry on such an element $\check{I}_n$ by the choices of ε_1 and ε_2 in the construction of the conjugation map ϕ_n as well as the positioning of the elements $\check{I}_n$. Here the "inner rotation map" is important.
Moreover, we compute that $\phi_n\left(\check{I}_n\right)$ lies in the "good area" of the map g_n. But the prior application of $D^{-1}_{\psi_n,\gamma_n}$ causes a translation of $\left(1 + \frac{1}{q_n^3} + ... + \frac{1}{q_n^{3+n-1}}\right)\cdot u \cdot 4\varepsilon_n$ with some $u \leq \frac{n^2}{2}$ in the r-coordinate. At first we observe that $D^{-1}_{\psi_n,\gamma_n} \circ \phi_n\left(\check{I}_n\right)$ is still contained in the same definition section of g_n by our choice of $j_1^{(2+3+...+(3+k-1))}$. Thus we compare the caused translation with an $\frac{\varepsilon}{b\cdot a} = \frac{16n^2\cdot\varepsilon_n}{n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}\cdot[nq_n^{\sigma_n}]}$-domain of the map $g_n = g_{a,b,\varepsilon,\delta}$ on the r-axis. In case of $2 + 3\cdot(k+1) + \frac{k\cdot(k+1)}{2} \geq 3 + n - 1$ the shifting is a multiple of such a domain and then $D^{-1}_{\psi_n,\gamma_n} \circ \phi_n\left(\check{I}_n\right)$ is still contained in the "good area" of g_n. In the other case we write $1 + \frac{1}{q_n^3} + ... + \frac{1}{q_n^{3+n-1}} = \frac{l}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}} + R$ with $l \in \mathbb{Z}$ and some rest term $R < \frac{2}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}+1}}$. Since $\frac{l}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}}\cdot u\cdot 4\varepsilon_n$ is a multiple of $\frac{\varepsilon}{b\cdot a}$ we consider $\frac{2}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}+1}}\cdot u\cdot 4\varepsilon_n$. We have

$$\begin{aligned} n\cdot u\cdot[nq_n^{\sigma_n}]\cdot\frac{1}{2n^2\cdot\varepsilon_n} &\leq n\cdot\frac{n^2}{2}\cdot n\cdot q_n^{\sigma_n}\cdot 2\cdot n^9\cdot q_{n-1}^{5\cdot r(n-1)+1} \\ =& n^{13}\cdot q_n^{4\cdot r(n-1)}\cdot q_{n-1}^{5\cdot r(n-1)+1} = n^{13}\cdot q_{n-1}^{9\cdot r(n-1)+1} < q_n \end{aligned}$$

by our assumptions on the numbers q_n and σ_n in section 3.2. So this deviation is bounded by

$$\begin{aligned} \frac{2}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}+1}}\cdot u\cdot 4\varepsilon_n &< \frac{2}{q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}}\cdot\frac{2n^2\cdot\varepsilon_n}{n\cdot u\cdot[nq_n^{\sigma_n}]}\cdot u\cdot 4\varepsilon_n \\ &= \varepsilon_n\cdot\frac{16n^2\cdot\varepsilon_n}{[nq_n^{\sigma_n}]\cdot n\cdot q_n^{2+3\cdot(k+1)+\frac{k\cdot(k+1)}{2}}}. \end{aligned}$$

Then $D^{-1}_{\psi_n,\gamma_n} \circ \phi_n\left(\check{I}_n\right)$ is still contained in the "good area" of g_n.
Thus h_n acts as an isometry on the elements of the partition ζ_n. □

This Lemma implies that $h_n^{-1}|_{h_n\left(D^{-1}_{\psi_n,\gamma_n}(\hat{I}_n)\right)}$ is an isometry as well.
In the following we construct the f-invariant measurable Riemannian metric. This construction parallels the approach in [GK00], section 4.8.. Therefor we put $\omega_n := \left(H_n^{-1}\right)^* \omega_0$. Each ω_n is a smooth Riemannian metric because it is the pullback of a smooth metric via a $C^\infty(M)$-diffeomorphism. Since $R^*_{\alpha_{n+1}}\omega_0 = \omega_0$ the metric ω_n is f_n-invariant:

$$\begin{aligned} f_n^*\omega_n &= \left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}\right)^* \left(H_n^{-1}\right)^* \omega_0 = \left(H_n^{-1}\right)^* R^*_{\alpha_{n+1}} H_n^* \left(H_n^{-1}\right)^* \omega_0 = \left(H_n^{-1}\right)^* R^*_{\alpha_{n+1}}\omega_0 \\ &= \left(H_n^{-1}\right)^* \omega_0 = \omega_n \end{aligned}$$

With the succeeding Lemmas we show that the limit $\omega_\infty := \lim_{n\to\infty}\omega_n$ exists μ-almost everywhere and is the aimed f-invariant Riemannian metric.

Lemma 5.2. *The sequence $(\omega_n)_{n\in\mathbb{N}}$ converges μ-a.e. to a limit ω_∞*

Proof. For every $N \in \mathbb{N}$ we have for every $k > 0$:

$$\omega_{N+k} = \left(H_{N+k}^{-1}\right)^* \omega_0 = \left(h_{N+k}^{-1} \circ ... \circ h_{N+1}^{-1} \circ H_N^{-1}\right)^* \omega_0 = \left(H_N^{-1}\right)^* \left(h_{N+k}^{-1} \circ ... \circ h_{N+1}^{-1}\right)^* \omega_0$$

Since the elements of the partition ζ_n cover M except a set of measure at most $\frac{3}{n^2}$ by Remark 2.6 Lemma 5.1 shows that ω_{N+k} coincides with $\omega_N = \left(H_N^{-1}\right)^*\omega_0$ on a set of measure at least $1 - \sum_{n=N+1}^{\infty} \frac{3}{n^2}$. As this measure approaches 1 for $N \to \infty$ the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges on a set of full measure. □

Lemma 5.3. *The limit ω_∞ is a measurable Riemannian metric.*

Proof. The limit ω_∞ is a measurable map because it is the pointwise limit of the smooth metrics ω_n, which in particular are measurable. By the same reasoning $\omega_\infty|_p$ is symmetric for μ-almost every $p \in M$. Furthermore ω_∞ is positive definite, because ω_n is positive definite for every $n \in \mathbb{N}$ and ω_∞ coincides with ω_N on $T_1M \otimes T_1M$ minus a set of measure at most $\sum_{n=N+1}^{\infty} \frac{3}{n^2}$. Since this is true for every $N \in \mathbb{N}$ ω_∞ is positive definite on a set of full measure. □

Remark 5.4. In the proof of the subsequent Lemma we will need Egoroff's theorem (for example [Ha65], §21, Theorem A): Let (N, d) denote a separable metric space. Given a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of N-valued measurable functions on a measure space (X, Σ, μ) and a measurable subset $A \subseteq X$, $\mu(A) < \infty$, such that $(\varphi_n)_{n\in\mathbb{N}}$ converges μ-a.e. on A to a limit function φ. Then for every $\varepsilon > 0$ there exists a measurable subset $B \subset A$ such that $\mu(B) < \varepsilon$ and $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ uniformly on $A \setminus B$.

Lemma 5.5. *ω_∞ is f-invariant, i.e. $f^*\omega_\infty = \omega_\infty$ μ-a.e..*

Proof. By Lemma 5.2 the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges in the C$^\infty$-topology pointwise almost everywhere. Hence we obtain using Egoroff's theorem: For every $\delta > 0$ there is a set $C_\delta \subseteq M$ such that $\mu(M \setminus C_\delta) < \delta$ and the convergence $\omega_n \to \omega_0$ is uniform on C_δ.
The function f was constructed as the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ in the C$^\infty$-topology. Thus $\tilde{f}_n := f_n^{-1} \circ f \to id$ in the C$^\infty$-topology. Since M is compact this convergence is uniform, too.
Furthermore the smoothness of f implies: $f^*\omega_\infty = f^* \lim_{n\to\infty}\omega_n = \lim_{n\to\infty} f^*\omega_n$. Hereby we compute on C_δ: $f^*\omega_\infty = \lim_{n\to\infty}\left(\left(f_n\tilde{f}_n\right)^*\omega_n\right) = \lim_{n\to\infty}\left(\tilde{f}_n^* f_n^*\omega_n\right) = \lim_{n\to\infty}\tilde{f}_n^*\omega_n = \omega_\infty$, where we used the uniform convergence on C_δ in the last step. As this holds on every set C_δ with $\delta > 0$ it also holds on the set $\bigcup_{\delta>0} C_\delta$. This is a set of full measure and therefore the claim follows. □

Hence the aimed f-invariant measurable Riemannian metric ω_∞ is constructed.

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